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Applications of Homogeneous Dynamics to Quadratic Forms

(동질 동역학의 이차 형식들로의 응용)

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Applications of Homogeneous Dynamics to Quadratic Forms

A dissertation
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Abstract

Applications of Homogeneous Dynamics to Quadratic Forms

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We study the geometry of quadratic forms using equidistribution theorems in homogeneous dynamics. First we study the mean square limit of exponential sums associated to a rational ellipsoid of arbitrary center. We obtain a lower bound for arbitrary center and that lower bound turns out to be the upper bound as well for ellipsoids with the center of certain diophantine type(see theorem 1.0.4). This result generalizes a work of Marklof.

The second topic is the quantitative Oppenheim conjecture for S-arithmetic quadratic forms. For an arbitrary open set I in $\mathbb{Q}_\mathsf{S}$, we show that the number of S-integral vectors of norm at most T, whose values of an irrational quadratic form are Q in I, is asymptotically $\mathsf{c}(\mathsf{Q},\mathrm{I})\mathsf{T}^{n-2}$ as T goes to infinity. This is a generalization of a work of Eskin-Margulis-Mozes for real case.

Key words: homogeneous dynamics, Jacobi theta sum, quantitative Oppenheim conjecture, equidistribution of unbounded functions.
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Contents

Chapter 1

Introduction

In this thesis, we treat two different applications of homogeneous dynamics to problems in number theory related to quadratic forms.

I. Distribution of Integral Lattice Points in an Ellipsoid with a Diophantine Center

The famous Gauss circle problem is to determine the number $N(r)$ of integral vectors in $\mathbb{R}^2$ with the Euclidean norm at most r. By considering a unit area rectangle attached to each integral vector, we can easily deduce that the asymptotic limit of $N(r)$ when r goes to infinity is the volume of the circle with radius r :

$$\lim_{r\to\infty} \frac{N(r)}{r^2} = \pi < \infty.$$

It still remains to obtain the first error term $E(r)$ between $N(r)$ and πr^2.

$$N(r) = \pi r^2 + E(r).$$

One can see that $E(r) = O(r)$ since by the above argument of unit rectangles, $E(r)$ is less than the area of δ-neighborhood of a circle with radius r, for some appropriate $\delta > 0$. Indeed, Gauss showed that $|E(r)| \leq 2\sqrt{2}\pi r$.([7]) In other words, if we let

$$|E(r)| = Cr^x \quad \text{for some } x \in \mathbb{R},$$

he showed that $x \leq 1$. In 1915, Hardy and Landau independently offered the lower bound by proving that([8],[13])

$$|E(r)| \neq o(r^{1/2}(\log r)^{1/4}).$$

Hardy suggests that $|E(r)| = O(r^{1/2+\epsilon})$ for any positive $\epsilon > 0$. However it remains unsolved. The current optimal upper bound of x is $131/208$ provided by Huxley.([10])

Recently, Kang and Sobolev treated a weaker statement about the error term([11]) as follows.

Let $\left[N(\overrightarrow{\alpha})\right](r)$ be the number of integral vectors whose distance from a vector $\overrightarrow{\alpha} \in \mathbb{R}^n$ is at most r and $\left[E(\overrightarrow{\alpha})\right](r)$ be the error term $\left[N(\overrightarrow{\alpha})\right](r) - \text{vol}(B_r)$.

Theorem 1.0.1. (Kang-Sobolev)[11, Theorem 1.1] Let $n \geq 2$.

$$\int_{-\infty}^{\infty} \frac{\left[E(\overrightarrow{\alpha})\right](r)}{r^{(n-1)/2}} \omega_R(r)dr \to 0 \quad \text{as } R \to \infty, \tag{1.1}$$

where $\omega \in C_0^\infty(\mathbb{R})$ is a nonnegative function such that $\omega(r) = 0$ for all $r \leq r_0$ with some $r_0 > 0$ and $\int \omega(r)dr = 1$ and $\omega_R(r) = \omega(r/R)/R$.

In the proof of Theorem 1.0.1, they used the results of Marklof [18] about the mean square value of the exponential sums defined by

$$\left[r\left(\overrightarrow{\alpha}\right)\right](d) = \sum_{\substack{\overrightarrow{m} \in \mathbb{Z}^n \\ \|\overrightarrow{m}\|^2 = d}} \exp(2\pi i \overrightarrow{m} \cdot \overrightarrow{\alpha}), \quad d \in \mathbb{N}.$$

for $\overrightarrow{\alpha} \in \mathbb{R}^n$ satisfying the diophantine condition:

Definition 1.0.2. A vector $\overrightarrow{\alpha} \in \mathbb{R}^n$ is said to be *of diophantine type* κ, if there exists a constant $C > 0$ such that

$$\left|\overrightarrow{\alpha} - \frac{\overrightarrow{m}}{q}\right| > \frac{C}{q^\kappa} \tag{1.2}$$

for all $\overrightarrow{m} \in \mathbb{Z}^n$ and $q \in \mathbb{N}$.

The smallest possible value of κ is $\frac{n+1}{n}$ and in this case, $\overrightarrow{\alpha}$ is called badly approximable. Note that the set of vectors having diophantine type κ is of full Lebesgue measure([18], [24, Theorme 6G]). For such a vector $\overrightarrow{\alpha} \in \mathbb{R}^n$, Marklof proved the following theorem.

Theorem 1.0.3. (Marklof)[18, Theorem 1.1] Assume $\overrightarrow{\alpha} \in \mathbb{R}^n$, $n \geq 2$ is such that the components of $(\overrightarrow{\alpha}, 1) \in \mathbb{R}^{n+1}$ are linearly independent over $\mathbb{Q}$. Then

$$\liminf_{M\to\infty} \frac{1}{M^{n/2}} \sum_{d=0}^{M} \left|\left[r(\overrightarrow{\alpha})\right](d)\right|^2 \geq \mathrm{vol}(B_n),$$

where B_n is the unit ball in $\mathbb{R}^n$. If, in addition, $\overrightarrow{\alpha}$ is diophantine of type $\kappa < (n-1)/(n-2)$, then

$$\lim_{M\to\infty} \frac{1}{M^{n/2}} \sum_{d=0}^{M} \left|\left[r(\overrightarrow{\alpha})\right](d)\right|^2 = \mathrm{vol}(B_n).$$

In the first part of the thesis, we generalize the above Marklof's theorem to rational ellipsoids. Let $\overrightarrow{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ be an integral vector with $a_i > 0$ for all i. Define a quadratic form $Q_{\overrightarrow{a}}$ on $\mathbb{R}^n$ by

$$Q_{\overrightarrow{a}}(x_1, \ldots, x_n) = a_1 x_1^2 + \cdots + a_n x_n^2, \tag{1.3}$$

and the corresponding ellipsoid $\Omega_{\overrightarrow{a}}$ by

$$\Omega_{\overrightarrow{a}} = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_1 x_1^2 + \cdots + a_n x_n^2 \leq 1\right\}. \tag{1.4}$$

It is easy to see that the number of the intersection of lattice points and the dilation $R\Omega_{\overrightarrow{a}}$ asymptotically goes to the volume of $R\Omega_{\overrightarrow{a}}$ as R goes to infinity. We also define a rational ellipsoid centered at $\overrightarrow{\alpha} \in \mathbb{R}^n$ by

$$\Omega_{\overrightarrow{a}}(\overrightarrow{\alpha}) = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_1(x_1 - \alpha_1)^2 + \cdots + a_n(x_n - \alpha_n)^2 \leq 1\right\} \tag{1.5}$$

and its dilation

$$R\Omega_{\vec{a}}(\vec{\alpha}) = \left\{(x_1, \ldots, x_n) \in \mathbb{R}^n : a_1(x_1-\alpha_1)^2 + \cdots + a_n(x_n-\alpha_n)^2 \le R^2\right\}. \quad (1.6)$$

Let $\left[r_{\vec{a}}\left(\vec{\alpha}\right)\right]$ be the exponential sum corresponding to a vector $\vec{a}$ by

$$\left[r_{\vec{a}}\left(\vec{\alpha}\right)\right](d) = \sum_{\substack{\vec{m} \in \mathbb{Z}^n \\ \|\vec{m}\|_{\vec{a}}^2 = d}} \exp\left(2\pi i \vec{m}\cdot\vec{\alpha}\right), \quad d \in \mathbb{N}, \quad (1.7)$$

where $\|\vec{m}\|_{\vec{a}} = \left(a_1 m_1^2 + \cdots + a_n m_n^2\right)^{1/2}$. Then we show the following theorem.

Theorem 1.0.4. (H.-Kang-Kim-Lim)[9] Let $\vec{\alpha} \in \mathbb{R}^n$, $n \ge 2$ be a vector such that $\left(\vec{\alpha}, 1\right) \in \mathbb{R}^{n+1}$ are linearly independent over $\mathbb{Q}$. For a positive integral vector $\vec{a} \in \mathbb{N}^n$, we have

$$\liminf_{N\to\infty} \frac{1}{N^{n/2}} \sum_{d=1}^{N} \left|\left[r_{\vec{a}}\left(\vec{\alpha}\right)\right](d)\right|^2 \ge \mathrm{vol}\left(\Omega_{\vec{a}}\left(\vec{\alpha}\right)\right). \quad (1.8)$$

If we add the condition that $\vec{\alpha}$ is of diophantine type $\kappa < \frac{n-1}{n-2}$, then

$$\lim_{N\to\infty} \frac{1}{N^{n/2}} \sum_{d=1}^{N} \left|\left[r_{\vec{a}}\left(\vec{\alpha}\right)\right](d)\right|^2 = \mathrm{vol}\left(\Omega_{\vec{a}}\left(\vec{\alpha}\right)\right), \quad (1.9)$$

where vol is the Lebesgue measure on $\mathbb{R}^n$.

If we let $\left[E_{\vec{a}}(\vec{\alpha})\right](r) = \left[N_{\vec{a}}(\vec{\alpha})\right](r) - \mathrm{vol}(r\Omega_{\vec{a}})$, as a corollary of Theorem 1.0.4, we can show the following:

Theorem 1.0.5. (H.-Kang-Kim-Lim)[9] Let $n \ge 2$ and $\vec{\alpha}, \vec{a} \in \mathbb{R}^n$.

$$\int_{-\infty}^{\infty} \frac{\left[E_{\vec{a}}(\vec{\alpha})\right](r)}{r^{(n-1)/2}} \omega_R(r)dr \to 0 \quad \text{as } R \to \infty, \quad (1.10)$$

where $\omega \in C_0^\infty(\mathbb{R})$ is a nonnegative function such that $\omega(r) = 0$ for all $r \le r_0$ with some $r_0 > 0$ and $\int \omega(r)dr = 1$ and $\omega_R(r) = \omega(r/R)/R$.

II. Quantitative Oppenheim Conjecture for S-arithmetic case

Another classical application of homogeneous dynamics to number theory is Oppenheim conjecture. Consider a nondegenerate indefinite quadratic form Q of rank n at least 3, defined over $\mathbb{R}$. We are interested in the distribution of $Q(\mathbb{Z}^n)$ in $\mathbb{R}$. If Q is the multiple of a rational quadratic form, then obviously the set $Q(\mathbb{Z}^n)$ is discrete. The conjecture by Oppenheim says that the quadratic values $Q(\mathbb{Z}^n)$ of integral vectors are dense in $\mathbb{R}$ when Q is irrational, that is, it is not the multiple of a rational quadratic form([20]). The earlier accomplishments were by Birch-Davenport-Ridout for n at least 21([1], [4], [23]) and by Davenport-Heibronn for n at least 5([2]) using analytic number theory. In 1987, Margulis solved the Oppenheim conjecture for n at least 3([19]) using ergodic theoretical methods, mainly the equidistribution theorem of a unipotent flow on homogeneous spaces and Dani-Margulis nondivergence theorem.

Few years later, Dani-Margulis([3]) and Eskin-Margulis-Mozes([6]) showed the following quantitative version of Oppenheim conjecture.

Let us define $\rho : \mathbb{S}^{n-1} \to \mathbb{R}_{>0}$, $\Omega = \{\overrightarrow{v} \in \mathbb{R}^n : \|\overrightarrow{v}\| < \rho(\overrightarrow{v}/\|\overrightarrow{v}\|)\}$ and $T\Omega = \{\overrightarrow{v} \in \mathbb{R}^n : \overrightarrow{v}/T \in \Omega\}$. for any interval $(a,b) \subset \mathbb{R}$, let

$$V^Q_{(a,b)} = \{\overrightarrow{x} \in \mathbb{R}^n : a < Q(\overrightarrow{x}) < b\}.$$

Theorem 1.0.6. (Dani-Margulis, Eskin-Margulis-Mozes)[6, Theorem 2.1] Let Q be an indefinite quadratic form of signature (p,q), with $p \geq 3$ and $q \geq 1$. Suppose Q is not proportional to a rational form. Then for any interval (a,b), as $T \to \infty$,

$$|\mathbb{Z}^n \cap V^Q_{(a,b)} \cap T\Omega| \sim \lambda_{Q,\Omega}(b-a)T^{n-2},$$

where $n = p+q$ and a constant $\lambda_{Q,\Omega}$ is given by

$$\lambda_{Q,\Omega} = \lim_{T\to\infty} \frac{\mathrm{vol}(V^Q_{(a,b)} \cap T\Omega)}{(b-a)T^{n-2}}.$$

Now let us describe our result. Consider the number field $K = \mathbb{Q}$ and let S be a finite set of places over K containing the archimedean place ∞ and $S_f = S \setminus \{\infty\}$ consisting of finite places. Recall that a finite(or a nonarchimedean) place over $\mathbb{Q}$ is a p-adic norm for some prime p, given by $\|x\|_p = p^{-n}$, where $x = p^n\ a/b$, $\gcd(a, p) = \gcd(b, p) = 1$.

Recall that an Ad-unipotent one-parameter subgroup U_ν in G_ν is a subgroup

$$U_\nu = \{u_\nu(z) : z \in \mathbb{Q}_\nu\}$$

such that all $u_\nu(z)$ are Ad-unipotent and $u_\nu(z_1 + z_2) = u_\nu(z_1)u_\nu(z_2)$, $z_1, z_2 \in \mathbb{Q}_p$([22]). Let $U \subset SL_n(\mathbb{Q}_S)$ be the product $U_0 \times \prod_{p \in S_f} U_p$ of Ad-unipotent one-parameter unipotent subgroups for each place $\nu \in S$. For $t = (t_0, t_1, \ldots, t_s) \in \mathbb{R} \times \mathbb{Z}^s$, let

$$U(t) = \{u_0(z_0) \times \prod_{p \in S_f} u_p(z_p) : 0 < z_0 < e^t,\ |z_p|_p < p^{t_p}\} \subseteq U.$$

In a sequel, we say that $x \in G/\Gamma$ is generic for U if for every bounded continuous function f on G/Γ,

$$\lim_{t \to \infty} \frac{1}{e^t \prod_{p \in S_f} p^{t_p}} \int_{U(t)} f(z(t)x)dz = \int_{G/\Gamma} f dg, \tag{1.11}$$

where dz and dg are normalized Haar measure on $\mathbb{Q}_S$ and G/Γ, respectively. It is a fact that if a quadratic form $Q = Q_0^g$ consists of irrational quadratic forms Q^ν, $\nu \in S$, then $g\Gamma$ is generic for U.

Let ρ be the product $\prod_{\nu \in S} \rho_\nu$ of continuous positive functions defined as

(a) $\rho_0 : \mathbb{S}^{n-1} = \{\overrightarrow{v}_0 \in \mathbb{R}^n : \|\overrightarrow{v}_0\| = 1\} \to \mathbb{R}_{>0}$;

(b) $\rho_i : \mathbb{U}^n_{p_i} = \{\overrightarrow{v}_i \in \mathbb{Q}^n_{p_i} : \|\overrightarrow{v}_i\| = 1\} \to \{p^z : z \in \mathbb{Z}\}$ satisfying that

$$\rho_i(u\overrightarrow{v}_i) = \rho_i(\overrightarrow{v}_i) \tag{1.12}$$

for any $\overrightarrow{v}_i \in \mathbb{U}^n_{p_i}$ and $u \in \mathcal{U}_{p_i} = \mathbb{Z}_{p_i} - p_i\mathbb{Z}_{p_i}$, where $S_f = \{p_1, \ldots, p_s\}$.

Define

$$\Omega = \{\overrightarrow{v} \in \mathbb{Q}_S^n : \|\overrightarrow{v}_0\| < \rho_0(\overrightarrow{v}_0/\|\overrightarrow{v}_0\|) \text{ and } \|\overrightarrow{v}_i\| \leq \rho_i(\|\overrightarrow{v}_i\|\overrightarrow{v}_i), 1 \leq i \leq s\} \quad (1.13)$$

and for $T = (T_0, T_1, \ldots, T_s) \in \mathbb{R}_{>0}^n$,

$$T\Omega = \{\overrightarrow{v} \in \mathbb{Q}_S^n : (T_0^{-1}\overrightarrow{v}_0, T_1\overrightarrow{v}_1, \ldots, T_s\overrightarrow{v}_s) \in \Omega\}.$$

For $a_0, b_0 \in \mathbb{R}$, $a_p \in \mathbb{Q}_p$ and $b_p \in \mathbb{Z}$, $p \in S_f$, we will denote $I(a,b)$ is a subset of $\mathbb{Q}_S^n$ of the form

$$I(a,b) = (a_0, b_0) \times \prod_{p \in S_f} (a_p + p^{-b_p}\mathbb{Z}_p)$$

and

$$V_{I(a,b)}^Q = \{\overrightarrow{v} \in \mathbb{Q}_S^n : Q(\overrightarrow{v}) \in I(a,b)\}.$$

Our main theorem is about the asymptotics of the number $|\mathbb{Z}_S^n \cap V_{I(a,b)}^Q \cap T\Omega|$ as $T \to \infty$, proving the quantitative Oppenheim conjecture for S-arithmetic case, suggested by G. Margulis.

Theorem 1.0.7. Let Q be a nondegenerate isotropic quadratic form such that

(a) every quadratic form Q^ν is irrational,

(b) The signature of Q^0 is (p,q), with $p \geq 3$ and $q \geq 1$.

Then for any interval $I(a,b)$, as $T_i \to \infty$ for each i,

$$|\mathbb{Z}_S^n \cap V_{I(a,b)}^Q \cap T\Omega| \sim c(Q, \Omega, I(a,b))|T|^{n-2},$$

where $|T| = \prod_{i=0}^{s} T_i$.

Part I

Distribution of Integral Lattice Points in an Ellipsoid with a Diophantine Center

Chapter 2

Two Representations and Jacobi Theta Sums

Let $\mathcal{S}(\mathbb{R}^n)$ be the Schwartz space, that is, the space of rapidly decreasing functions on $\mathbb{R}^n$. We want to define two representations, namely the Schrödinger representation of a Heisenberg group $\mathbb{H}(\mathbb{R}^n)$ and the projective Shale-Weil representation of a symplectic group $\mathrm{Sp}_n(\mathbb{R})$ acting on the Schwartz space $\mathcal{S}(\mathbb{R}^n)(\subset \mathfrak{L}^2(\mathbb{R}^n))$.

2.1 Schrödinger representation

Let ω be the standard symplectic form on $\mathbb{R}^{2n}$ given by:

$$\omega\left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, \begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix},\right) = \vec{x} \cdot \vec{y}' - \vec{y} \cdot \vec{x}', \tag{2.1}$$

where $\overrightarrow{x}, \overrightarrow{x}', \overrightarrow{y}, \overrightarrow{y}' \in \mathbb{R}^n$ and $\cdot$ is the standard inner product in $\mathbb{R}^n$. Define the Heisenberg group $\mathbb{H}(\mathbb{R}^n)$ as a group $\mathbb{R}^{2n} \ltimes \mathbb{R}$ with multiplication law:

$$\begin{aligned}
&\left(\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}, t\right)\left(\begin{pmatrix} \overrightarrow{x}' \\ \overrightarrow{y}' \end{pmatrix}, t'\right) \\
&\qquad = \left(\begin{pmatrix} \overrightarrow{x} + \overrightarrow{x}' \\ \overrightarrow{y} + \overrightarrow{y}' \end{pmatrix}, t + t' + \frac{\overrightarrow{x} \cdot \overrightarrow{y}' - \overrightarrow{y} \cdot \overrightarrow{x}'}{2}\right) \\
&\qquad = \left(\begin{pmatrix} \overrightarrow{x} + \overrightarrow{x}' \\ \overrightarrow{y} + \overrightarrow{y}' \end{pmatrix}, t + t' + \frac{1}{2}\omega\left(\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}, \begin{pmatrix} \overrightarrow{x}' \\ \overrightarrow{y}' \end{pmatrix}\right)\right). \qquad (2.2)
\end{aligned}$$

The Lagrangian subspace of $\mathbb{R}^{2n}$ with respect to ω is a subspace $\mathfrak{l}$ satisfying

$$\mathfrak{l} = \mathfrak{l}^{\perp} = \left\{\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix} \in \mathbb{R}^{2n} : \omega\left(\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}, \begin{pmatrix} \overrightarrow{x}' \\ \overrightarrow{y}' \end{pmatrix}\right) = 0 \ \text{ for every } \begin{pmatrix} \overrightarrow{x}' \\ \overrightarrow{y}' \end{pmatrix} \in \mathfrak{l}\right\}.$$

For any Lagrangian subspace $\mathfrak{l}$, there is a Lagrangian subspace $\mathfrak{l}'$ such that $\mathbb{R}^{2n} = \mathfrak{l} \oplus \mathfrak{l}'$. For example,

$$\begin{aligned}
\mathfrak{l} &= \left\{\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{0} \end{pmatrix} : \overrightarrow{x} \in \mathbb{R}^n\right\} = \mathbb{R}^n(\overrightarrow{x}) \text{ and} \\
\mathfrak{l}' &= \left\{\begin{pmatrix} \overrightarrow{0} \\ \overrightarrow{y} \end{pmatrix} : \overrightarrow{y} \in \mathbb{R}^n\right\} = \mathbb{R}^n(\overrightarrow{y}).
\end{aligned}$$

Note that any element of $\mathbb{H}(\mathbb{R}^n)$ can be written as follows;

$$\left(\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}, t\right) = \left(\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{0} \end{pmatrix}, 0\right)\left(\begin{pmatrix} \overrightarrow{0} \\ \overrightarrow{y} \end{pmatrix}, 0\right)\left(\begin{pmatrix} \overrightarrow{0} \\ \overrightarrow{0} \end{pmatrix}, t - \frac{\overrightarrow{x} \cdot \overrightarrow{y}}{2}\right).$$

Fix $\mathfrak{l} = \mathbb{R}^n(\overrightarrow{x})$ and $\mathfrak{l}' = \mathbb{R}^n(\overrightarrow{y})$. Let $L \subset \mathbb{H}(\mathbb{R}^n)$ be the normal subgroup defined by

$$L = \left\{\left(\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{0} \end{pmatrix}, t\right) : \overrightarrow{x} \in \mathbb{R}^n, t \in \mathbb{R}\right\}.$$

Then the quotient space $\mathbb{H}(\mathbb{R}^n)/L$ is isomorphic to $\mathbb{R}^n = \mathbb{R}^n(\overrightarrow{y})$.

Let f be a function on $\mathbb{H}(\mathbb{R}^n)$ defined by

$$f\left(\left(\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}, t\right)\right) = e^{2\pi i t}.$$

Then the restriction of f on L is a character on L. Using the function f, define

$$\left\{\phi \in \mathfrak{L}^2(\mathbb{H}(\mathbb{R}^n)) : \phi(gh) = f(h)^{-1}\phi(g) \text{ for any } g \in \mathbb{H}(\mathbb{R}^n) \text{ and } h \in L\right\}. \tag{2.3}$$

Since ϕ in (2.3) is determined by values on $\mathbb{H}(\mathbb{R}^n)/L \cong \mathbb{R}^n(\overrightarrow{y})$ and any $\mathfrak{L}^2$-function ψ on $\mathbb{R}^n(\overrightarrow{y})$ induces an element of the set (2.3), we can identify the set (2.3) with $\mathfrak{L}^2(\mathbb{R}^n(\overrightarrow{y}))$.

The Schrödinger representation $W = W^f(\mathfrak{l})$ is the induced representation of $\mathbb{H}(\mathbb{R}^n)$ by the character f of the normal subgroup L on $\mathfrak{L}^2(\mathbb{R}^n(\overrightarrow{y}))$ using the above identification: For such a function ϕ and $g_0 \in \mathrm{Sp}_n(\mathbb{R})$,

$$\left(W(g_0)\phi\right)(g) = \phi(g_0^{-1}g).$$

In particular, for $g_0 = \left(\begin{pmatrix} \overrightarrow{x}_0 \\ \overrightarrow{y}_0 \end{pmatrix}, t_0\right)$ and $\phi \in \mathfrak{L}^2(\mathbb{R}^n) \left(= \mathfrak{L}^2(\mathbb{R}^n(\overrightarrow{y}))\right)$,

$$\left(W(g_0)\phi\right)(\overrightarrow{y}) = \exp\left(2\pi i\left(t - \frac{\overrightarrow{x}_0 \cdot \overrightarrow{y}_0}{2}\right)\right)\exp\left(2\pi i\,\overrightarrow{x}_0 \cdot \overrightarrow{y}\right)\phi\left(\overrightarrow{y} - \overrightarrow{y}_0\right).$$

It is obvious that the Schrödinger representation W is a unitary representation on a Hilbert space $\mathfrak{L}^2(\mathbb{R}^n)$.

2.2 Shale-Weil representation

The symplectic group $\mathrm{Sp}_n(\mathbb{R})$ is the collection of $2n$ by $2n$ invertible matrices preserving the symplectic form ω. That is,

$$\begin{aligned} \mathrm{Sp}_n(\mathbb{R}) &= \left\{ g \in \mathfrak{M}_{2n}(\mathbb{R}) :^t g \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathfrak{M}_{2n}(\mathbb{R}) : \begin{array}{c} {}^tCA =^t AC \\ {}^tAD -^t CB = \mathrm{Id} \\ {}^tDB =^t BD \end{array} \right\} \end{aligned} \tag{2.4}$$

Define an action of $\mathrm{Sp}_n(\mathbb{R})$ on $\mathbb{H}(\mathbb{R}^n)$ by

$$g.\left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t \right) = \left(g \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t \right),$$

where $g \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}$ is a matrix multiplication.

A projective Shale-Weil representation of $\mathrm{Sp}_n(\mathbb{R})$ can be defined based on the following theorem.

Theorem 2.2.1. (Stone-von Neumann Theorem)

1. $W^f(l)$ is an irreducible representation of $\mathbb{H}(\mathbb{R}^n)$;

2. Every unitary representation T of $\mathbb{H}(\mathbb{R}^n)$ on a Hilbert space $\mathcal{H}$ satisfying that

$$T\left(\begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}, t \right) = \mathrm{Id}_{\mathcal{H}}$$

is a multiple of $W^f(l)$.

We skip the proof of Stone-von Neumann Theorem. For details, see section 1.3 in [14].

For any element $g \in \mathrm{Sp}_n(\mathbb{R})$, define a map W_g from $\mathbb{H}(\mathbb{R}^n)$ to $\mathrm{Aut}\left(\mathfrak{L}^2(\mathbb{R}^n)\right)$ by

$$W_g\left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right) = W\left(g\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right).$$

Since g preserves the symplectic form ω, for any $\left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right)$ and $\left(\begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix}, t'\right)$ in $\mathbb{H}(\mathbb{R}^n)$,

$$\begin{aligned}
& W_g\left(\left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right)\left(\begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix}, t'\right)\right) \\
= \;& W_g\left(\begin{pmatrix} \vec{x}+\vec{x}' \\ \vec{y}+\vec{y}' \end{pmatrix}, t+t'+\frac{1}{2}\omega\left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, \begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix}\right)\right) \\
= \;& W\left(g\begin{pmatrix} \vec{x}+\vec{x}' \\ \vec{y}+\vec{y}' \end{pmatrix}, t+t'+\frac{1}{2}\omega\left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, \begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix}\right)\right) \\
= \;& W\left(g\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} + g\begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix}, t+t'+\frac{1}{2}\omega\left(g\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, g\begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix}\right)\right) \\
= \;& W\left(\left(g\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right)\left(g\begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix}, t'\right)\right) \\
= \;& W\left(\left(g\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right)\right) W\left(\left(g\begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix}, t'\right)\right) \\
= \;& W_g\left(\left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right)\right) W_g\left(\left(\begin{pmatrix} \vec{x}' \\ \vec{y}' \end{pmatrix}, t'\right)\right).
\end{aligned}$$

Therefore W_g is a (unitary) representation of $\mathbb{H}(\mathbb{R}^n)$ on $\mathfrak{L}^2(\mathbb{R}^n)$ satisfying

$$W_g\left(\begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}, t\right)\phi = W\left(\begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}, t\right)\phi = e^{2\pi i t}\,\phi, \;\; \phi \in \mathfrak{L}^2(\mathbb{R}^n).$$

By Stone-von Neumann theorem, W_g and W are unitary equivalent, that is, there exists a unitary operator $R(g) : \mathfrak{L}^2(\mathbb{R}^n) \longrightarrow \mathfrak{L}^2(\mathbb{R}^n)$ such that

$$R(g)W\left(\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}, t\right)R(g)^{-1} = W_g\left(\begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}, t\right)$$

Note that the operator $R(g)$ is well-defined up to scalar multiplication of unit modulus. Although it does not satisfy that $R(g_1g_2) = R(g_1)R(g_1)$, there is a constant $c(g_1, g_2)$ of unit absolute value such that

$$R(g_1g_2) = c(g_1, g_2)R(g_1)R(g_2).$$

We will call R a projective Shale-Weil representation and c a cocycle of a representation R. In our case, when $\mathfrak{l} = \mathbb{R}^n\left(\overrightarrow{x}\right)$, for $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp_n(\mathbb{R})$, $R(g)$ is given by;

$$\left(R(g)\phi\right)\left(\overrightarrow{y}\right) = \int_{\overrightarrow{\xi} \in \mathbb{R}^n/\ker^t C} \phi\left({}^tA\overrightarrow{y} +^t C\overrightarrow{\xi}\right) e^{\pi i (A^tB\overrightarrow{y})\cdot\overrightarrow{y}} e^{2\pi i ({}^tB\overrightarrow{y})\cdot({}^tC\overrightarrow{\xi})} e^{\pi i (D^tC\overrightarrow{\xi})\cdot\overrightarrow{\xi}}.$$

For convenience, we want to introduce explicit expressions of $R(g)$ for 3 types of matrices which we will use later.

1. Let $g \in Sp_n(\mathbb{R})$ is of the form $\begin{pmatrix} A & 0 \\ 0 & {}^tA^{-1} \end{pmatrix}$, then

$$\left(R(g)\phi\right)\left(\overrightarrow{y}\right) = |\det A|^{1/2}\phi\left({}^tA\overrightarrow{y}\right).$$

2. Let $g \in Sp_n(\mathbb{R})$ is of the form $\begin{pmatrix} I_n & B \\ 0 & I_n \end{pmatrix}$ with $B =^t B$, then

$$\left(R(g)\phi\right)\left(\overrightarrow{y}\right) = e^{\pi i (B\overrightarrow{y})\cdot\overrightarrow{y}}\phi(\overrightarrow{y}).$$

3. Let $g \in \mathrm{Sp}_n(\mathbb{R})$ is of the form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with C invertible, then

$$(\mathrm{R}(g)\phi)\,(\overrightarrow{y}) = \int_{\mathbb{R}^n} \phi(\overrightarrow{\xi})e^{\pi i\left((C^{-1}D\overrightarrow{\xi}\cdot\overrightarrow{\xi}+(AC^{-1}\overrightarrow{y})\cdot\overrightarrow{y}-2(C^{-1}Y)\cdot\overrightarrow{\xi}\right)}.$$

2.3 Maslov index and the cocycle of R

Recall that for a given triple of Lagrangian subspaces l_1, l_2 and l_3, the Maslov index $\tau(l_1, l_2, l_3)$ is defined as follow.

Definition 2.3.1. (Maslov index) For any triple of Lagrangian planes l_1, l_2 and l_3, the Maslov index $\tau(l_1, l_2, l_3)$ is given by $p-q$, where (p, q) is a signature of the quadratic form $Q(v_1 + v_2 + v_3)$ on the vector space $l_1 \oplus l_2 \oplus l_3$ defined by:

$$Q(v_1 + v_2 + v_3) = \omega(v_1, v_2) + \omega(v_2, v_3) + \omega(v_3, v_1). \tag{2.5}$$

We will denote $p - q$ by $\mathrm{sign}(Q)$.

Since any element g of $\mathrm{Sp}_n(\mathbb{R})$ preserves the symplectic form ω, if l is a Lagrangian space, then gl is also a Lagrangian space.

Lemma 2.3.2. Suppose that Lagrangian subspaces l_1 and l_3 are transverse. Let p_{13} and p_{31} be projections to l_1 and l_3 respectively, based on the decomposition $\mathbb{R}^{2n} = l_1 \oplus l_3$. If we define the quadratic form Q' on l_2 by

$$Q'(v) = \omega(p_{13}v, p_{31}v) = \omega(v, p_{31}v) = \omega(p_{13}v, v),$$

then $\tau(l_1, l_2, l_3)$ coincides with the signature of the quadratic form Q'.

It turns out that the cocycle $c(g_1, g_2)$ of the Shale-Weil representation is given by the exponential of Maslov index:

$$c(g_1, g_2) = \exp\left(-\frac{\pi}{4} i\tau(\mathbb{R}^n(\overrightarrow{x}), g_1\mathbb{R}^n(\overrightarrow{x}), g_1 g_2\mathbb{R}^n(\overrightarrow{x}))\right).$$

In the following special case, we can easily calculate the cocycle $c(g_1, g_2)$.

Proposition 2.3.3. When $g_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}$ and $g_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}$ in $\mathrm{Sp}_n(\mathbb{R})$ such that C_1 and C_2 are invertible, let us denote

$$g_1 g_2 = \begin{pmatrix} A_3 & B_3 \\ C_3 & D_3 \end{pmatrix}.$$

Then $c(g_1, g_2) = \exp\left(-\frac{\pi}{4} i \operatorname{sign}\left(C_1^{-1} C_3 C_2^{-1}\right)\right)$.

Proof. First note that $C_1^{-1} C_3 C_2^{-1}$ is a symmetric matrix. Since $C_3 = C_1 A_2 + D_1 C_2$ and g_1 and g_2 satisfy the equation (2.4),

$$\begin{aligned} {}^t\left(C_1^{-1} C_3 C_2^{-1}\right) &= {}^t\left(A_2\, C_2^{-1} + C_1^{-1} D_1\right) \\ &= {}^t C_2^{-1}\, {}^t A_2 + {}^t D_1\, {}^t C_1^{-1} D_1 \\ &= A_2\, C_2^{-1} + C_1^{-1} D_1 \\ &= C_1^{-1} C_3\, C_2^{-1}. \end{aligned}$$

We will show that $\tau(\mathfrak{l}, g_1 \mathfrak{l}, g_1 g_2 \mathfrak{l}) = \operatorname{sign}\left(C_1^{-1} C_3 C_2^{-1}\right)$.

Since g_1 preserves ω,

$$\tau(\mathfrak{l}, g_1 \mathfrak{l}, g_1 g_2 \mathfrak{l}) = \tau(g_1^{-1} \mathfrak{l}, \mathfrak{l}, g_2 \mathfrak{l}) = -\tau(g_1^{-1} \mathfrak{l}, g_2 \mathfrak{l}, \mathfrak{l}).$$

For any $v = \begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix} \in \mathfrak{l}$,

$$v \in g_1^{-1} \mathfrak{l} \cap \mathfrak{l} \quad \Leftrightarrow \quad g_1 v = \begin{pmatrix} A_1 \vec{x} \\ C_1 \vec{x} \end{pmatrix} \in \mathfrak{l} \tag{2.6}$$

$$\Leftrightarrow \quad C_1 \vec{x} = \vec{0} \Leftrightarrow \vec{x} = \vec{0}, \tag{2.7}$$

since C_1 is invertible. Hence $g_1^{-1} \mathfrak{l}$ and $\mathfrak{l}$ are transverse. By Lemma 2.3.2, $\tau(g_1^{-1} \mathfrak{l}, g_2 \mathfrak{l}, \mathfrak{l})$ is the signature of $\omega\left(p_{13}(g_2 v), p_{31}(g_2 v)\right)$ with the correspondence $\mathfrak{l}_1 = g_1^{-1} \mathfrak{l}$, $\mathfrak{l}_2 = g_2 \mathfrak{l}$ and $\mathfrak{l}_3 = \mathfrak{l}$.

We also compute that for $v \in \mathfrak{l}$

$$\begin{aligned} g_2 v &= p_{13}(g_2 v) + p_{31}(g_2 v) \\ &= \begin{pmatrix} -{}^tD_1\,{}^tC_1^{-1}\,C_2\overrightarrow{x} \\ C_2\overrightarrow{x} \end{pmatrix} + \begin{pmatrix} A_2\overrightarrow{x} +^t D_1\,{}^tC_1^{-1}\,C_2\overrightarrow{x} \\ \overrightarrow{0} \end{pmatrix}. \end{aligned}$$

Therefore

$$\begin{aligned} \omega\left(p_{13}(g_2 v), p_{31}(g_2 v)\right) &= \omega\left(\begin{pmatrix} -{}^tD_1\,{}^tC_1^{1}\,C_2\overrightarrow{x} \\ C_2\overrightarrow{x} \end{pmatrix}, \begin{pmatrix} A_2\overrightarrow{x} +^t D_1\,{}^tC_1^{-1}\,C_2\overrightarrow{x} \\ \overrightarrow{0} \end{pmatrix}\right) \\ &= -C_2\overrightarrow{x} \cdot \left(A_2\overrightarrow{x} +^t D_a\,{}^tC_1^{-1}C_2\overrightarrow{x}\right) \\ &= -\begin{pmatrix} A_2\overrightarrow{x} & C_2\overrightarrow{x} \end{pmatrix} \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & A_2\,C_2^{-1} +^t D_1\,{}^tC_1^{-1} \end{pmatrix} \begin{pmatrix} A_2\overrightarrow{x} \\ C_2\overrightarrow{x} \end{pmatrix}, \end{aligned}$$

hence $\tau(\mathfrak{l}, g_1\mathfrak{l}, g_1 g_2\mathfrak{l}) = \operatorname{sign}\left(A_2\,C_2^{-1} +^t D_1\,{}^tC_1^{-1}\right) = \operatorname{sign}\left(C_1^{-1}C_3C_2^{-1}\right)$. □

2.4 The subgroup $SL_2(\mathbb{R})^n$ and notations

From now on, we will concentrate our interests on a subgroup $SL_2(\mathbb{R})^n$ of $Sp_n(\mathbb{R})$ given by

$$SL_2(\mathbb{R})^n = \left\{ \begin{pmatrix} a_1 & & & b_1 & & \\ & \ddots & & & \ddots & \\ & & a_n & & & b_n \\ c_1 & & & d_1 & & \\ & \ddots & & & \ddots & \\ & & c_n & & & d_n \end{pmatrix} : \begin{array}{c} a_i d_i - b_i c_i = 1, \\ 1 \le i \le n \end{array} \right\}$$

which is isomorphic to a product of n copies of $SL_2(\mathbb{R})$. For convenience, we will denote an element of $SL_2(\mathbb{R})^n$ by $\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}_{j=1}^n$ or simply $\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}_j$.

By restricting the Shale-Weil representation to $SL_2(\mathbb{R})^n$, we can decompose

$\mathbb{R}^{2n} = \oplus_{j=1}^{n}\mathbb{R}^2\left(\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)$, where each $\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}$ acts on $\mathbb{R}^2\left(\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)$ componentwise.

Proposition 2.4.1. If $g = \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}_j$, $g' = \begin{pmatrix} a_j' & b_j' \\ c_j' & d_j' \end{pmatrix}_j$ and $g'' = \begin{pmatrix} a_j'' & b_j'' \\ c_j'' & d_j'' \end{pmatrix}_j$ in $\mathrm{SL}_2(\mathbb{R})^n$ with $gg' = g''$, then the corresponding $c(g, g')$ is

$$c(g, g') = \exp\left(-\frac{\pi}{4} i \sum_{j=1}^{n} \mathrm{sign}(c_j c_j' c_j'')\right),$$

where $\mathrm{sign}(x) = x/|x|$ for $x \in \mathbb{R}\backslash\{0\}$ and $\mathrm{sign}(0) = 0$.

Proof. By the above argument, the Maslov index $\tau\left(\mathbb{R}^n(\overrightarrow{x}), g_1\mathbb{R}^n(\overrightarrow{x}), g_1g_2\mathbb{R}^n(\overrightarrow{x})\right)$ is of the form

$$\begin{aligned}
&\tau\left(\mathbb{R}^n(\overrightarrow{x}), g_1\mathbb{R}^n(\overrightarrow{x}), g_1g_2\mathbb{R}^n(\overrightarrow{x})\right) \\
&\qquad = \sum_{j=1}^{n} \tau\left(\mathbb{R}(x_j), \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}\mathbb{R}(x_j), \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}\begin{pmatrix} a_j' & b_j' \\ c_j' & d_j' \end{pmatrix}\mathbb{R}(x_j)\right),
\end{aligned}$$

where $\mathbb{R}(x_j) = \left\{\begin{pmatrix} x_j \\ 0 \end{pmatrix} : x_j \in \mathbb{R}\right\} \subset \mathbb{R}^2$. Thus it suffices to prove when $n = 1$. In case that both c and c' are not zero, it follows from proposition[numbering?]. If $c = 0$, that is, $g = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}$, then it is easy to check that $g\mathbb{R}(x) = \mathbb{R}(x)$ so that $\omega(v_1, v_2) \equiv 0$ and $\mathrm{sign}\left[\omega(v_2, v_3)\right] = -\mathrm{sign}\left[\omega(v_3, v_1)\right]$, where $v_1 \in \mathbb{R}(x)$, $v_2 \in g\mathbb{R}(x)$ and $v_3 \in gg'\mathbb{R}(x)$. Similarly, if $c' = 0$ then we have that $g\mathbb{R}(x) = gg'\mathbb{R}(x)$, $\omega(v_2, v_3) \equiv 0$ and $\mathrm{sign}\left[\omega(v_1, v_2)\right] = -\mathrm{sign}\left[\omega(v_3, v_1)\right]$. Therefore the required equality holds. □

Using the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$, any element $\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}_j$ can be uniquely expressed as follow.

$$\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}_j = \begin{pmatrix} 1 & u_j \\ 0 & 1 \end{pmatrix}_j \begin{pmatrix} v_j^{1/2} & 0 \\ 0 & v_j^{-1/2} \end{pmatrix}_j \begin{pmatrix} \cos\phi_j & -\sin\phi_j \\ \sin\phi_j & \cos\phi_j \end{pmatrix}_j$$
$$= (z_j, \phi_j)_j \,,$$

where $z_j = u_j + iv_j \in \mathbb{H}$, a upper-half plane and $\phi_j \in [0, 2\pi)$. We will use the notation $(z_j, \phi_j)_j$ for an element $SL_2(\mathbb{R})^n$ as well. Note that the left multiplication of $SL_2(\mathbb{R})^n$ is given by:

$$\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}_j (z_j, \phi_j)_j = \left(\frac{a_j z_j + b_j}{c_j z_j + d_j}, \phi_j + \arg(c_j z_j + d_j) \mod 2\pi \right)_j .$$

The Shale-Weil representation of $SL_2(\mathbb{R})^n$ with this coordinate $(z_j, \phi_j)_j$ is given by the following: for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$[R(z_j, \phi_j)_j f]\,(\overrightarrow{\xi}) = (v_1 \cdots v_n)^{1/4} \exp(\pi i \sum_j u_j \xi_j^2)\, [R(i, \phi_j)_j f] \left(v_1^{1/2} \xi_1, \ldots, v_n^{1/2} \xi_n \right).$$

Note that when $\phi_j = \frac{\pi}{2}$, $j = 1, \ldots, n$, the map $R\left(i, \frac{\pi}{2}\right)_j$ on $\mathcal{S}(\mathbb{R}^n)$ coincides the Fourier transformation:

$$\left[R\left(i, \frac{\pi}{2}\right)_j f \right] (\overrightarrow{\xi}) = \int_{\mathbb{R}^n} e^{-2\pi i \overrightarrow{x} \cdot \overrightarrow{\xi}} f(\overrightarrow{x}) d\overrightarrow{x}.$$

Actually the way in which Shale-Weil representation is defined does not guarantee of continuity. Indeed, for $f \in \mathcal{S}(\mathbb{R}^n)$,

$$\lim_{\phi_1 \to 0^\pm} \cdots \lim_{\phi_n \to 0^\pm} [R(i, \phi_j)_j f]\,(\overrightarrow{\xi}) = e^{\pm \frac{k\pi}{4} i} f(\overrightarrow{\xi}),$$

for some integer $-n \le k \le n$. Thus in order to obtain the continuity, let us define a new representation $\tilde{R}$ by putting

$$\tilde{R}(z_j, \phi_j)_j = \exp\left(-\frac{\pi}{4} i (\sigma_{\phi_1} + \cdots + \sigma_{\phi_n}) \right) R\,(z_j, \phi_j)_j \,,$$

where $\sigma_{\phi_k} = 2\nu$ if $\phi_k = \nu\pi$ and $\sigma_{\phi_k} = 2\nu + 1$ if $\nu\pi < \phi_k < (\nu+1)\pi$ for some integer ν.

2.5 Jacobi's theta sum

Definition 2.5.1. For $f \in \mathcal{S}(\mathbb{R}^n)$, the Jacobi's theta sum $\Theta_f = \Theta_f\left((z_j, \phi_j)_j; \left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right)\right)$ is a function on a semi-direct product $\mathrm{SL}_2(\mathbb{R})^n \ltimes \mathbb{H}(\mathbb{R}^n)$ defined by

$$\Theta_f\left((z_j, \phi_j)_j; \left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right)\right) = \sum_{\vec{m} \in \mathbb{Z}^n} \left[W\left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right) \tilde{R}\,(z_j, \phi_j)_j\, f \right](\vec{m})$$
$$= (v_1 \cdots v_k)^{\frac{1}{4}} \exp\left(2\pi i t - \pi i \vec{x} \cdot \vec{y}\right) \times$$
$$\sum_{\vec{m} \in \mathbb{Z}^k} f_{\phi_J}((m_1 - y_1)v_1^{\frac{1}{2}}, \ldots, (m_k - y_k)v_k^{\frac{1}{2}}) \exp\left(\pi i \sum (m_j - y_j)^2 u_j + 2\pi i \vec{m} \cdot \vec{x}\right),$$

where $f_{\phi_J} = \left[\tilde{R}\,(i, \phi_j)_j\, f\right]$. The summation is well-defined since f is rapidly decreasing.

We are interested in a product $\Theta_f \overline{\Theta_g}$ of a Jacobi's theta sum Θ_f and the complex conjugation of a Jacobi's theta sum Θ_g for f, $g \in \mathcal{S}(\mathbb{R}^n)$. Then we can get rid of the variable t in $\Theta_f \overline{\Theta_g}$ and we can think of the Jacobi's theta sum as a function defined on a semi-direct product $G^n = \mathrm{SL}_2(\mathbb{R})^n \ltimes \mathbb{R}^{2n}$ from the beginning. Note that the group G^n is isomorphic to a product of n copies of $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$.

Proposition 2.5.2. Let Γ^n be a discrete subgroup of G^n defined by;

$$\Gamma^n = \left\{ \left(\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}; \begin{pmatrix} \frac{1}{2} a_j b_j \\ \frac{1}{2} c_j d_j \end{pmatrix} + \begin{pmatrix} m_j \\ n_j \end{pmatrix} \right)_j : \begin{array}{c} a_j, b_j, c_j, d_j, m_j, n_j \in \mathbb{Z} \\ a_j d_j - b_j c_j = 1 \end{array} \right\}.$$

Then the left action of Γ^n on G^n is properly discontinuous. Moreover, a fundamental domain of Γ^n is given by;

$$\mathfrak{F}_{\Gamma^n} = \prod_{j=1}^{n} \left(\mathfrak{F}_{\mathrm{SL}_2(\mathbb{Z})} \times \{\phi_j \in [0,\pi]\} \times \left\{ \begin{pmatrix} x_j \\ y_j \end{pmatrix} \in [-\frac{1}{2}, \frac{1}{2})^2 \right\} \right),$$

where $\mathfrak{F}_{\mathrm{SL}(2,\mathbb{Z})} = \{z = u + iv \in \mathbb{H}^2 : u \in [-\frac{1}{2}, \frac{1}{2}), |z| > 1\}$ is the fundamental domain in $\mathbb{H}^2$ of the modular group $\mathrm{SL}_2(\mathbb{Z})$.

Proof. It is enough to show that the proposition holds for $n = 1$.

The coordinate $(z = u + iv, \phi)$ which comes from the Iwasawa decomposition of $\mathrm{SL}_2(\mathbb{R})$ shows that we can think of a quotient space $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathrm{SL}_2(\mathbb{R})$ as the unit tangent bundle of $\mathrm{SL}_2(\mathbb{Z}) \setminus \mathbb{H}^2$. We already know that the fundamental domain of the action $\mathrm{SL}_2(\mathbb{Z})$ on $\mathbb{H}^2$ is $\{z = u + iv \in \mathbb{H}^2 : u \in [-\frac{1}{2}, \frac{1}{2}), |z| > 1\}$. The variable ϕ ranges between 0 and π because of an element $-I_2$. For given $\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}; \begin{pmatrix} \xi \\ \eta \end{pmatrix} \right) \in \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^{2n}$, choose $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathfrak{F}_{\mathrm{SL}_2(\mathbb{Z})}.$$

And then we can find integers m and n uniquely such that

$$\begin{pmatrix} \frac{1}{2}ab \\ \frac{1}{2}cd \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \xi \\ \eta \end{pmatrix} + \begin{pmatrix} m \\ n \end{pmatrix}$$

lies in $[-1/2, 1/2)$. □

Note that Γ^n is generated by three kinds of matrices below;

$$g_n = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} \vec{n}_1 \\ \vec{n}_2 \end{pmatrix} \right) \text{ for } \vec{n}_1, \vec{n}_2 \in \mathbb{Z}^n,$$

$$g_{f_1} = \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix} \right),$$

$$g_{f_2} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}\right),$$
$$\vdots$$
$$g_{f_n} = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}; \begin{pmatrix} \vec{0} \\ \vec{0} \end{pmatrix}\right),$$

$$u_1 = \left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} \vec{s}_1 \\ \vec{0} \end{pmatrix}\right),$$
$$u_2 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} \vec{s}_2 \\ \vec{0} \end{pmatrix}\right),$$
$$\vdots$$
$$u_n = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \ldots, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} \vec{s}_k \\ \vec{0} \end{pmatrix}\right),$$

where $\vec{s}_j$ is an element of $\mathbb{R}^n$ whose entries are 0 except that j-th whose entry is $\frac{1}{2}$.

Proposition 2.5.3. For $f, g \in \mathcal{S}(\mathbb{R}^n)$, $\Theta_f\overline{\Theta_g}$ is invariant under the left action of Γ^n.

Proof. In the proof, we put the variable t for the convenience of computation. The claim is that the action of Γ^n affects theta sums only on exponentials which will be vanishing in $\Theta_f\overline{\Theta_g}$. It suffices to show this for generators of Γ^n.

i) $g_{f_l}.\Theta_f\left((z_j, \phi_j); \begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j = \exp(-\frac{i\pi}{4})\Theta_f\left((z_j, \phi_j); \begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j.$

For $f \in \mathcal{S}(\mathbb{R}^k)$,

$$\tilde{R}(g_{f_l})f(m_1, \ldots, m_k) = \int_{\mathbb{R}} e^{-2\pi i m_l x_l} f(m_1, \ldots, x_l, \ldots, m_n)dx_l$$

which is the l-th partial Fourier transformation of $\mathbb{R}^n$.

Since $\left[W\left(\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix},t\right)\tilde{R}\left((z_j,\phi_j)_j\right)f\right]\in\mathcal{S}(\mathbb{R}^n)$ if $f\in\mathcal{S}(\mathbb{R}^n)$, by the Poisson summation formula,

$$\begin{aligned}
&\sum_{\vec{m}\in\mathbb{Z}^n}\tilde{R}(g_{f_j})\left[W\left(\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix},t\right)\tilde{R}\left((z_j,\phi_j)_j\right)f\right](\vec{m})\\
&\qquad=\sum_{m_1,\ldots,\widehat{m_l},\ldots,m_n\in\mathbb{Z}}\ \sum_{m_l\in\mathbb{Z}}\tilde{R}(g_{f_l})\left[W\left(\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix},t\right)\tilde{R}\left((z_j,\phi_j)_j\right)f\right](\vec{m})\\
&\qquad=\sum_{m_1,\ldots,\widehat{m_l},\ldots,m_n\in\mathbb{Z}}\ \sum_{m_l\in\mathbb{Z}}\left[W\left(\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix},t\right)\tilde{R}\left((z_j,\phi_j)_j\right)f\right](\vec{m})\\
&\qquad=\sum_{\vec{m}\in\mathbb{Z}^n}\left[W\left(\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix},t\right)\tilde{R}\left((z_j,\phi_j)_j\right)f\right](\vec{m}).
\end{aligned}$$

On the other hand, using proposition 2.4, we obtain that

$$\begin{aligned}
&\tilde{R}(g_{f_l})\tilde{R}\left((z_j,\phi_j);\begin{pmatrix}x_j\\ y_j\end{pmatrix}\right)_j=\tilde{R}(g_{f_l})W\left(\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix},t\right)\tilde{R}\left((z_j,0)_j\right)\tilde{R}\left((i,\phi_j)_j\right)\\
&=W\left(g_{f_l}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix},t\right)\tilde{R}(g_{f_l})\tilde{R}\left((z_j,0)_j\right)\tilde{R}\left((i,\phi_j)_j\right)\\
&=W\left(g_{f_l}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix},t\right)\tilde{R}\big((z_1,0),\ldots,(-\frac{1}{z_l},\arg z_l),\ldots,(z_n,0)\big)\tilde{R}\left((i,\phi_j)_j\right)\\
&=W\left(g_{f_l}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix},t\right)\tilde{R}\big((z_1,0),\ldots,(-\frac{1}{z_l},0),\ldots,(z_n,0)\big)\\
&\qquad\qquad\circ\tilde{R}\big((i,0),\ldots,(i,\arg z_l),\ldots,(i,0)\big)\tilde{R}\left((i,\phi_j)_j\right)\\
&=W\left(g_{f_l}\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix},t\right)\tilde{R}\big((z_1,0),\ldots,(-\frac{1}{z_l},0),\ldots,(z_n,0)\big)\\
&\qquad\qquad\circ\exp\left(\frac{\pi i}{4}\right)\tilde{R}\big((i,\phi_1),\ldots,(i,\phi_l+\arg z_l),\ldots,(i,\phi_n)\big)
\end{aligned}$$

$$= \exp\left(\frac{\pi i}{4}\right) W\left(g_{f_l}\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right) \tilde{R}\big((z_1,\phi_1),\ldots,(-\frac{1}{z_l},\phi_l+\arg z_l),\ldots,(z_n,\phi_n)\big),$$

where the exponential term comes from the fact that $0 < \arg \tau_l < \pi$ for $\tau_l \in \mathbb{H}^2$. Since

$$g_{f_l}\left((z_j,\phi_j)_j ; \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right) = \left((z_1,\phi_1),\ldots,(-\frac{1}{z_l},\phi_l+\arg z_l),\ldots,(z_n,\phi_n); g_{f_l}\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right),$$

we see that

$$g_{f_l}.\Theta_f\left((z_j,\phi_j);\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j = \sum_{\vec{m}\in\mathbb{Z}^n}[g_{f_l}.\tilde{R}\left((z_j,\phi_j);\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j f](\vec{m})$$
$$= \sum_{\mathbf{m}\in\mathbb{Z}^k} \exp(-\frac{\pi i}{4})[R(g_{f_l})\tilde{R}\left((z_j,\phi_j);\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j f](\vec{m})$$
$$= \exp(-\frac{\pi i}{4})\sum_{\vec{m}\in\mathbb{Z}^n}[\tilde{R}\left((z_j,\phi_j);\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j f](\vec{m}) = \exp(-\frac{\pi i}{4})\Theta_f\left((z_j,\phi_j);\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j.$$

ii) $u_l.\Theta_f\left((z_j,\phi_j);\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j = \exp\left(-\frac{\pi i}{2}y_l\right)\Theta_f\left((z_j,\phi_j);\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j.$

For $f \in \mathcal{S}(\mathbb{R}^n)$, since $m_l(m_l+1) \in 2\mathbb{Z}$,

$$\begin{aligned}\tilde{R}(u_l)f(\vec{m}) &= W\left(\begin{pmatrix} \vec{s}_l \\ \vec{0} \end{pmatrix}, 0\right)\tilde{R}\big((i,0),\ldots,(i+1,0),\ldots,(i,0)\big)f(\vec{m}) \\ &= \exp(2\pi i\,\vec{s}_l\cdot\vec{m})\exp(\pi i\, m_l^2)f(\vec{m}) \\ &= \exp(\pi i\, m_l(m_l+1))f(\vec{m}) = f(\vec{m}).\end{aligned}$$

Replacing f by $\left[W\left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t\right)\tilde{R}\left((z_j,\phi_j)_j\right) f\right]$, it follows that

$$\sum_{\vec{m}\in\mathbb{Z}^n} \tilde{R}(u_l)\left[W\left(\left(\begin{array}{c}\vec{x}\\ \vec{y}\end{array}\right), t\right)\tilde{R}\left((z_j,\phi_j)_j\right)f\right](\vec{m})$$
$$= \sum_{\vec{m}\in\mathbb{Z}^n}\left[W\left(\left(\begin{array}{c}\vec{x}\\ \vec{y}\end{array}\right), t\right)\tilde{R}\left((z_j,\phi_j)_j\right)f\right](\vec{m}).$$

Since

$$\left[W\left(\left(\begin{array}{c}\vec{x}\\ \vec{y}\end{array}\right), t+s\right)\tilde{R}\left((\tau_j,\phi_j)_j\right)f\right](\vec{m})$$
$$= \exp^{2\pi i\, s}\left[W\left(\left(\begin{array}{c}\vec{x}\\ \vec{y}\end{array}\right), t+s\right)\tilde{R}\left((\tau_j,\phi_j)_j\right)f\right](\vec{m}) \text{ and}$$

$$\tilde{R}(u_l)\left[W\left(\left(\begin{array}{c}\vec{x}\\ \vec{y}\end{array}\right), t+s\right)\tilde{R}\left((\tau_j,\phi_j)_j\right)\right]$$
$$= W\left(\left(\left(\begin{array}{c}x_1\\ y_1\end{array}\right),\ldots,\left(\begin{array}{c}\frac{1}{2}\\ 0\end{array}\right)+\left(\begin{array}{cc}1&1\\ 0&1\end{array}\right)\left(\begin{array}{c}x_l\\ y_l\end{array}\right),\ldots,\left(\begin{array}{c}x_k\\ y_k\end{array}\right)\right), t+\frac{1}{4}y_l\right)$$
$$\circ\,\tilde{R}\big((z_1,\phi_1),\ldots,(z_l+1,\phi_l),\ldots,(z_n,\phi_n)\big),$$

we obtain

$$\tilde{R}(u_l)\Theta_f\left((z_j,\phi_j);\left(\begin{array}{c}x_j\\ y_j\end{array}\right)\right)_j = \exp\left(-\frac{\pi i}{2}y_l\right)\Theta_f\left(u_l\left((z_j,\phi_j);\left(\begin{array}{c}x_j\\ y_j\end{array}\right)\right)_j\right).$$

$$\text{iii)}\,g_n.\Theta_f\left((z_j,\phi_j);\left(\begin{array}{c}x_j\\ y_j\end{array}\right)\right)_j = \exp\left(-\pi i\left(\vec{n}_1\cdot\vec{n}_2\right)\right)\Theta_f\left((z_j,\phi_j);\left(\begin{array}{c}x_j\\ y_j\end{array}\right)\right)_j.$$

A similar calculation shows that since $\vec{n}_1 \in \mathbb{Z}^n$,

$$
\begin{aligned}
& g_n . \left[\tilde{R} \left((z_j, \phi_j)_j ; \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t \right) f \right] (\vec{m}) \\
&= \left[\tilde{R} \left((z_j, \phi_j)_j ; \begin{pmatrix} \vec{n}_1 \\ \vec{n}_2 \end{pmatrix} + \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t + \frac{1}{2}\omega \left(\begin{pmatrix} \vec{n}_1 \\ \vec{n}_2 \end{pmatrix}, \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \right) \right) f \right] (\vec{m}) \\
&= \left[W \left(\begin{pmatrix} \vec{n}_1 \\ \vec{n}_2 \end{pmatrix}, 0 \right) W \left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t \right) \tilde{R} (z_j, \phi_j)_j f \right] (\vec{m}) \\
&= \exp\left(-\pi i \vec{n}_1 \cdot \vec{n}_2\right) \exp\left(2\pi i \vec{n}_1 \cdot \vec{n}_2\right) \left[W \left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t \right) \tilde{R} (z_j, \phi_j)_j f \right] (\vec{m} - \vec{n}_2) \\
&= \exp\left(-\pi i \vec{n}_1 \cdot \vec{n}_2\right) \left[W \left(\begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix}, t \right) \tilde{R} (z_j, \phi_j)_j f \right] (\vec{m} - \vec{n}_2)
\end{aligned}
$$

Hence we obtain the desired equality since summation in the theta sum is accomplished over all integral vectors.

2.6 Relation between Jacobi's theta sums and the mean square value of exponential sums

In this subsection, we examine how we can express the mean square value of exponential sums by Jacobi's theta sums.

Denote $\|\vec{x}\|_{\vec{a}}$ for a positive integral vector $\vec{a} = (a_1, \ldots, a_n) \in \mathbb{Z}^n$ by

$$
\|\vec{x}\|_{\vec{a}} = \left(a_1 x_1^2 + \cdots + a_n x_n^2 \right)^{1/2}, \quad \vec{x} \in \mathbb{R}^n. \tag{2.8}
$$

Let us first consider the explicit form of $\Theta_f \overline{\Theta_g}$ for $f, g \in \mathcal{S}(\mathbb{R}^n)$ as follows;

$$\Theta_f\left((z_j,\phi_j);\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)\overline{\Theta_g\left((z_j,\phi_j);\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)} = (v_1 v_2\cdots v_n)^{\frac{1}{2}}$$
$$\times \sum_{\vec{m},\vec{n}\in\mathbb{Z}^n} f_\phi((m_1-y_1)v_1^{\frac{1}{2}},\ldots,(m_n-y_n)v_n^{\frac{1}{2}})\overline{g_\phi((m_1-y_1)v_1^{\frac{1}{2}},\ldots,(m_n-y_n)v_n^{\frac{1}{2}})}$$
$$\times \exp\pi i\left(\sum_{j=1}^n u_j(m_j-y_j)^2+2\vec{m}\cdot\vec{x}\right)\exp\pi i\left(-\left(\sum_{j=1}^n u_j(n_j-y_j)^2+2\vec{n}\cdot\vec{x}\right)\right),$$

where $f_\phi=\left[\tilde{R}\,(i,\phi_j)_j\, f\right]$ and $g_\phi=\left[\tilde{R}\,(i,\phi_j)_j\, g\right]$.

We will put $z_j=a_ju+ia_jv$, $\phi_j=0$ and $\vec{y}=\vec{0}$. Let $\mathbb{1}_{[0,1]}$ be a function on $\mathbb{R}^n$ defined by

$$\mathbb{1}_{[0,1]}(\vec{x})=\mathbb{1}_{[0,1]}\left(\|\vec{x}\|^2\right)=\begin{cases}1, & x_1^2+\cdots+x_n^2\le 1;\\ 0, & \text{otherwise,}\end{cases}$$

where $\|\cdot\|$ is a usual Euclidean norm of $\mathbb{R}^n$.

Take an non-decreasing sequence $(f_k)_{k=1}^\infty$ and a non-increasing sequence $(f'_k)_{k=1}^\infty$ in $\mathcal{S}(\mathbb{R}^n)$ such that $f_k\le\mathbb{1}_{[0,1]}\le f'_k$ and $|f'_k-f_k|_\infty\to 0$ as $k\to\infty$. Then the limit

$$\begin{aligned}&\lim_{k\to\infty}\Theta_{f_k}\overline{\Theta_{f_k}}\left(=\lim_{k\to\infty}\Theta_{f'_k}\overline{\Theta_{f'_k}}\right)\\ &=(a_1\cdots a_n)^{1/2}v^{n/2}\sum_{\vec{m},\vec{n}\in\mathbb{Z}^k}\mathbb{1}_{[0,1]}\left(\|\vec{m}\|_{\vec{a}}^2v\right)\mathbb{1}_{[0,1]}\left(\|\vec{n}\|_{\vec{a}}^2v\right)\times\\ &\qquad\exp\left(\pi iu\left(\|\vec{m}\|_{\vec{a}}^2-\|\vec{n}\|_{\vec{a}}^2\right)+2\pi i(\vec{m}-\vec{n})\cdot\vec{x}\right).\end{aligned}$$

By integrating both sides on $[0,2]$ in terms of u and using Lebesgue's dominated convergence theorem, we get that

$$\lim_{k\to\infty}\frac{1}{2}\int_0^2 \Theta_{f_k}\overline{\Theta_{f_k}}du = (a_1\cdots a_n)^{1/2}\nu^{n/2}\times \sum_{\substack{\vec{m},\,\vec{n}\in\mathbb{Z}^n,\\ \|\vec{m}\|^2_{\vec{a}}=\|\vec{n}\|^2_{\vec{a}}\le \nu^{-1}}} \exp\left(2\pi i\left(\vec{m}-\vec{n}\right)\cdot\vec{x}\right). \tag{2.9}$$

Recall that the exponential sum $\left[r_{\vec{a}}(\vec{\alpha})\right]$ is

$$\left[r_{\vec{a}}(\vec{\alpha})\right](d) = \sum_{\substack{\vec{m}\in\mathbb{Z}^n\\ \|\vec{m}\|^2_{\vec{a}}=d}} \exp\left(2\pi i\vec{m}\cdot\vec{\alpha}\right), \quad d\in\mathbb{N},$$

where $\vec{\alpha}\in\mathbb{R}^n$ is given. Then,

$$\begin{aligned}|\left[r_{\vec{a}}\left(\vec{\alpha}\right)\right](d)|^2 &= \sum_{\substack{\vec{m}\in\mathbb{Z}^n\\ \|\vec{m}\|^2_{\vec{a}}=d}}\ \sum_{\substack{\vec{n}\in\mathbb{Z}^k\\ \|\vec{n}\|^2_{\vec{a}}=d}} \exp\left(2\pi i\vec{m}\cdot\vec{\alpha}\right)\exp\left(-2\pi i\vec{n}\cdot\vec{\alpha}\right)\\ &= \sum_{\substack{\vec{m},\,\vec{n}\in\mathbb{Z}^n\\ \|\vec{m}\|_{\vec{a}}=\|\vec{n}\|_{\vec{a}}=d}} \exp\left(2\pi i(\vec{m}-\vec{n})\cdot\vec{\alpha}\right).\end{aligned} \tag{2.10}$$

By setting $M=\lfloor 1/\nu\rfloor$ and $\vec{x}=\vec{\alpha}$ in (2.9) and (2.10), we obtain that

$$\lim_{M\to\infty}\frac{1}{M^{n/2}}\sum_{d=1}^{M}\left[r_{\vec{a}}\left(\vec{\alpha}\right)\right](d)|^2 = (a_1\cdots a_n)^{-1/2}\lim_{\nu\to 0}\lim_{k\to\infty}\frac{1}{2}\int_0^2\Theta_{f_k}\overline{\Theta_{f_k}}du. \tag{2.11}$$

Chapter 3

Dynamics on $\mathrm{SL}_2(\mathbb{R})^n \ltimes \mathbb{R}^{2n}/\Gamma$

3.1 Equidistribution of closed orbits

Consider a subgroup

$$\left\{ g \in \mathrm{SL}_2(\mathbb{R})^n : \begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix}_j = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, j = 1, \ldots, n \right\} \subset \mathrm{SL}_2(\mathbb{R})^n$$

which is an embedded image of $\mathrm{SL}_2(\mathbb{R})$ and let's denote it by $\mathrm{SL}_2(\mathbb{R})$ as well by the abuse of notation. Thus the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathbb{R}^{2n}$ and a subgroup $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^{2n}$ of G^n are well defined.

For any positive integral vector $\overrightarrow{a} = (a_1, \ldots, a_n)$, since we can reparametrize u by $u' = \gcd(a_j)u$ and v by $v' = \gcd(a_j)v$, without loss of generality, we may assume that $\gcd(a_j)$ of a given integral vector $\overrightarrow{a} = (a_1, \ldots, a_n) \in \mathbb{N}^n$ is one. Denote

$$A = A_{\overrightarrow{a}} = \left(\begin{pmatrix} a_j^{1/2} & 0 \\ 0 & a_j^{-1/2} \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \end{pmatrix} \right)_j \in G^n.$$

For a given vector $\overrightarrow{a}$, we will consider a unipotent flow $\Psi^u_{\overrightarrow{a}}$ on $\Gamma^n \backslash G^n$ as right multiplication of the following one-parameter subgroup;

$$\Psi^u = \Psi^u_{\vec{a}} = \left(\begin{pmatrix} 1 & a_j u \\ 0 & 1 \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)_j \in G^n.$$

Also, we define the geodesic flow Φ^t of G^n as follow;

$$\Phi^t = \left(\begin{pmatrix} \exp(-\frac{t}{2}) & 0 \\ 0 & \exp(\frac{t}{2}) \end{pmatrix}; \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right)_j \in G^n.$$

Let $L_0 = SL_2(\mathbb{R}) \ltimes \mathbb{R}^{2n}$ and define $L_{\vec{a}} \subset SL_2(\mathbb{R})^n \ltimes \mathbb{R}^{2n}$ for each $\vec{a}$ by;

$$\begin{aligned} L_{\vec{a}} &= A_{\vec{a}} L A_{\vec{a}}^{-1} \\ &= \left\{ \left(\begin{pmatrix} a & a_j b \\ a_j^{-1} c & d \end{pmatrix}; \begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j : \begin{matrix} ad - bc = 1, \\ a, b, c, d, x_j, y_j \in \mathbb{R} \end{matrix} \right\}. \end{aligned} \tag{3.1}$$

Clearly, two flows $\Psi^u_{\vec{a}}$ and Φ^t are in $L_{\vec{a}}$ so that we can think of $\Psi^u_{\vec{a}}$ and Φ^t as unipotent flow and geodesic flow of $L_{\vec{a}}$ respectively as well. When $L_{\vec{a}} = L_0$, we will denote $\Psi^u_{\vec{a}}$ by Ψ^u_0. Then

$$\Psi^u_{\vec{a}} = A_{\vec{a}} \Psi^u_0 A_{\vec{a}}^{-1}.$$

For now, consider a lattice subgroup Γ^0 of $G^n = SL_2(\mathbb{R})^n \ltimes \mathbb{R}^{2n}$ defined by $\Gamma^0 = SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2n}$ and let $\Gamma^0_{\vec{a}}$ be a lattice subgroup $\Gamma^0_{\vec{a}}$ of $L_{\vec{a}}$ given by

$$\begin{aligned} \Gamma^0_{\vec{a}} &= \Gamma^n \cap L_{\vec{a}} \\ &= \left\{ \left(\begin{pmatrix} a & a_j b \\ \frac{1}{a_j} c & d \end{pmatrix}; \begin{pmatrix} m_j \\ n_j \end{pmatrix}\right)_j : \begin{matrix} ad - bc = 1, \\ a, b, d, m_j, n_j \in \mathbb{Z} \\ c \in \operatorname{lcm}(a_j)\mathbb{Z} \end{matrix} \right\}. \end{aligned} \tag{3.2}$$

For any finite-indexed subgroup Γ of Γ^0, $\Gamma_{\vec{a}} = \Gamma \cap L_{\vec{a}}$ is a finite-indexed subgroup of $\Gamma^0_{\vec{a}}$, hence $\Gamma_{\vec{a}}$ is also a lattice subgroup of $L_{\vec{a}}$.

We are concerned with the orbit of the unipotent flow $\Psi^u_{\vec{a}}$ at $\Gamma^0 g_0$ in $\Gamma^0 \setminus G^n$ for $g_0 \in G^n$ of the form

$$g_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_j ; \begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix} \right), \quad \vec{x} \in \mathbb{R}^n. \tag{3.3}$$

Let p be a projection from G^n to $\Gamma^0 \setminus G^n$. Then $p(L_{\vec{a}})$ is a closed submanifold of $\Gamma^0 \setminus G^n$ and we can take a homeomorphism between $\Gamma^0_{\vec{a}} \setminus L_{\vec{a}}$ and $p(L_{\vec{a}})$ given by $\Gamma^0 g \mapsto \Gamma^0_{\vec{a}} g$, $g \in L_{\vec{a}}$. Since $\Psi^u_{\vec{a}}$, Φ^t and g_0 are in $L_{\vec{a}}$, the orbit $\{\Gamma^0 g_0 \ \Psi^u_{\vec{a}} : u \in \mathbb{R}\}$ is in $p(L_{\vec{a}})$ and under the homeomorphism right above, we will denote it by $\left\{\Gamma^0_{\vec{a}} g_0 \Psi^u_{\vec{a}} : u \in \mathbb{R}\right\}$, too.

Since g_0 commutes with $\Psi^u_{\vec{a}}$ and $\Psi^1_{\vec{a}}$ is in $\Gamma^0_{\vec{a}}$,

$$\Gamma^0_{\vec{a}} g_0 \Psi^{u+1}_{\vec{a}} \Phi^t = \Gamma^0_{\vec{a}} g_0 \Psi^u_{\vec{a}} \Phi^t.$$

Thus the orbit $\left\{\Gamma^0_{\vec{a}} g_0 \Psi^u_{\vec{a}} \Phi^t : u \in \mathbb{R}\right\}$ on $\Gamma^0_{\vec{a}} \setminus L_{\vec{a}}$ can be described as the closed curve

$$\left\{\Gamma^0_{\vec{a}} g_0 \Psi^u_{\vec{a}} \Phi^t : u \in [0,1)\right\}. \tag{3.4}$$

If $\Gamma_{\vec{a}}$ is a finite-indexed subgroup of $\Gamma^0_{\vec{a}}$, $\Gamma_{\vec{a}} \setminus L_{\vec{a}}$ is a finite covering of $\Gamma^0_{\vec{a}} \setminus L_{\vec{a}}$. Consider all lifts of (3.4) on $\Gamma_{\vec{a}} \setminus L_{\vec{a}}$ which form in general a disjoint union of several closed curves. Note that each connected component is composed of the same number of lifts of (3.4).

Let $r = r(\Gamma_{\vec{a}})$ be the quotient of $[\Gamma^0_{\vec{a}} : \Gamma_{\vec{a}}]$ over the number of connected component of lifts. Then the closed curve

$$\left\{\Gamma_{\vec{a}} g_0 \Psi^u_{\vec{a}} \Phi^t : u \in [0,r)\right\} \tag{3.5}$$

represents an orbit $\left\{\Gamma_{\vec{a}} g_0 \Psi^u_{\vec{a}} \Phi^t : u \in \mathbb{R}\right\}$ on $\Gamma_{\vec{a}} \setminus L_{\vec{a}}$.

For a Γ^0-invariant function F defined on G^n, denote an induced function on $\Gamma^0 \setminus G^n$ by $\overline{F}$, a restriction on $L_{\vec{a}}$ by $F|_{L_{\vec{a}}}$, and an induced function on $\Gamma_{\vec{a}} \setminus L_{\vec{a}}$

by $\overline{F|_{L_{\vec{a}}}}$. For any $g \in L_{\vec{a}}$, $\overline{F}$ and $\overline{F|_{L_{\vec{a}}}}$ satisfy that

$$\overline{F}(\Gamma^0 g) = \overline{F|_{L_{\vec{a}}}}\,(\Gamma_{\vec{a}} g)\,. \tag{3.6}$$

Then we prove the following theorem using Theorem 3.1 by Marklof [18].

Theorem 3.1.1. Let F be a bounded continuous function on G^n which is invariant under Γ, a finite-indexed subgroup of $SL_2(\mathbb{Z})^n \ltimes \mathbb{Z}^{2n}$ and $g_0 = \left(\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_j ; \begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix}\right) \in G^n$, where the components of the vector $({}^t\vec{x}, 1) \in \mathbb{R}^{n+1}$ are $\mathbb{Q}$-linearly independent. Let $\vec{a}$ be a positive integral vector and h a piecewise continuous function on $\mathbb{R}/r\mathbb{Z}$, where $r = r(\Gamma_{\vec{a}})$. Then

$$\lim_{t\to\infty} \frac{1}{r}\int_0^r \overline{F} \circ \Phi^t \circ \Psi^u_{\vec{a}}(\Gamma g_0) h(u)\,du = \int_{\Gamma_{\mathbf{a}}\backslash L_{\mathbf{a}}} \overline{F|_{L_{\vec{a}}}}\, d\mu_{\vec{a}} \frac{1}{r}\int_0^r h(u)\,du, \tag{3.7}$$

where $\mu_{\vec{a}}$ is the normalized Haar measure of $\Gamma_{\vec{a}} \setminus L_{\vec{a}}$.

Proof. Theorem 3.1 [18] shows that (3.7) holds for $L_0 = SL_2(\mathbb{R}) \ltimes \mathbb{R}^{2n}$. To use this theorem, let us take a map $\varphi : L_{\vec{a}} \to L_0$ by;

$$\varphi\left(\left(\left(\begin{pmatrix} a & a_j b \\ \frac{1}{a_j}c & d \end{pmatrix}; \begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)\right)_j\right) = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} x_j \\ a_j y_j \end{pmatrix}\right)_j. \tag{3.8}$$

We claim that φ is a homomorphism. The group multiplication of G^n shows that

$$\begin{aligned}
&\left(\begin{pmatrix} a & a_j b \\ \frac{1}{a_j}c & d \end{pmatrix}; \begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j \left(\begin{pmatrix} a' & a_j b' \\ \frac{1}{a_j}c' & d' \end{pmatrix}; \begin{pmatrix} x'_j \\ y'_j \end{pmatrix}\right)_j \\
&= \left(\begin{pmatrix} a & a_j b \\ \frac{1}{a_j}c & d \end{pmatrix}\begin{pmatrix} a' & a_j b' \\ \frac{1}{a_j}c' & d' \end{pmatrix}; \begin{pmatrix} x_j \\ y_j \end{pmatrix} + \begin{pmatrix} a & a_j b \\ \frac{1}{a_j}c & d \end{pmatrix}\begin{pmatrix} x'_j \\ y'_j \end{pmatrix}\right)_j.
\end{aligned}$$

Hence

$$\varphi\left(\left(\left(\begin{array}{cc} a & a_j b \\ \frac{1}{a_j}c & d \end{array}\right);\left(\begin{array}{c} x_j \\ y_j \end{array}\right)\right)_j \left(\left(\begin{array}{cc} a' & a_j b' \\ \frac{1}{a_j}c' & d' \end{array}\right);\left(\begin{array}{c} x_j' \\ y_j' \end{array}\right)\right)\right)_j$$
$$= \left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right)\left(\begin{array}{cc} a' & b' \\ c' & d' \end{array}\right);\left(\begin{array}{c} x_j \\ a_j y_j \end{array}\right) + \left(\begin{array}{c} a x_j' + a_j b y_j' \\ c x_j' + a_j d y_j' \end{array}\right)\right)_j$$
$$= \left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right);\left(\begin{array}{c} x_j \\ a_j y_j \end{array}\right)\right)_j \left(\left(\begin{array}{cc} a' & b' \\ c' & d' \end{array}\right);\left(\begin{array}{c} x_j' \\ a_j y_j' \end{array}\right)\right)_j$$
$$= \varphi\left(\left(\left(\begin{array}{cc} a & a_j b \\ \frac{1}{a_j}c & d \end{array}\right);\left(\begin{array}{c} x_j \\ y_j \end{array}\right)\right)_j\right) \varphi\left(\left(\left(\begin{array}{cc} a' & a_j b' \\ \frac{1}{a_j}c' & d' \end{array}\right);\left(\begin{array}{c} x_j' \\ y_j' \end{array}\right)\right)_j\right).$$

Obviously φ is bijective so that the map φ is an isomorphism between $L_{\vec{a}}$ and L_0. Moreover, the image of $\Gamma^0_{\vec{a}}$

$$\varphi\left(\Gamma^0_{\vec{a}}\right) = \left\{\left(\left(\begin{array}{cc} a & b \\ c & d \end{array}\right);\left(\begin{array}{c} m_j \\ a_j n_j \end{array}\right)\right)_j : \begin{array}{c} ad - bc = 1, \\ a, b, d, m_j, n_j \in \mathbb{Z}, \\ c \in \mathrm{lcm}(a_j)\mathbb{Z} \end{array}\right\},$$

is a finite-indexed subgroup of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2n}$. It means that $\varphi(\Gamma_{\vec{a}})$ is also a finite-indexed subgroup of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2n}$.

Since φ is a group isomorphism, $F|_{L_{\vec{a}}} \circ \varphi^{-1}$ is a bounded continuous function on L_0 which is invariant under $\varphi(\Gamma_{\vec{a}})$.

By the theorem 3.1 of Marklof [18],

$$\lim_{t\to\infty} \frac{1}{r}\int_0^r (\overline{F|_{L_{\vec{a}}} \circ \varphi^{-1}}) \circ \Phi^t \circ \Psi^u_0(\varphi(\Gamma)g_0)h(u)du = \int_{\varphi(\Gamma_{\vec{a}})\backslash L_0} \overline{F|_{L_{\vec{a}}} \circ \varphi^{-1}} d\mu_0 \frac{1}{r}\int_0^r h(u)du,$$

where μ_0 is a normalized Haar measure of $\varphi(\Gamma_{\vec{a}})\backslash L_0$. After changing of variables, since φ^{-1} is also a group homomorphism,

$$\lim_{t\to\infty} \frac{1}{r}\int_0^r (\overline{F|_{L_{\vec{a}}}}) \circ \Phi^t \circ \Psi^u_{\vec{a}}(\Gamma g_0)h(u)du = \int_{\Gamma_{\vec{a}}\backslash L_{\vec{a}}} \overline{F|_{L_{\vec{a}}}} d\mu_{\vec{a}} \frac{1}{r}\int_0^r h(u)du,$$

where $\mu_{\overrightarrow{a}}$ is a normalized Haar measure of $\Gamma_{\overrightarrow{a}} \setminus L_{\overrightarrow{a}}$ which shows the theorem. $\square$

Concerning the isomorphism φ defined in (3.8) for each positive integral vector $\overrightarrow{a} = (a_1, \dots, a_n)$, we can extend the above theorem to dominated unbounded functions, which is the generalization of theorem 5.1 in [18]. Let us first introduce a dominated unbounded function on $L_0 = SL_2(\mathbb{R}) \ltimes \mathbb{R}^{2n}$ following [18].

For $f' \in \mathcal{S}(\mathbb{R}^n)$ and $R > 1$, define $F_R\left(z; \begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}\right)$ by

$$F_R\left(z; \begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}\right) = \sum_{\gamma \in \Gamma_\infty \setminus SL_2(\mathbb{Z})} \sum_{\overrightarrow{m} \in \mathbb{Z}^n} f'\left((\overrightarrow{y}_\gamma + \overrightarrow{m}) v_\gamma^{1/2}\right) v_\gamma^{n/2} \mathbb{1}_{[R,\infty)}(v_\gamma), \tag{3.9}$$

where $\Gamma_\infty = \left\{ \begin{pmatrix} 1 & m \\ 0 & 1 \end{pmatrix} : m \in \mathbb{Z} \right\}$, $v_\gamma = \mathrm{Im}\,(\gamma z) = \mathrm{Im}\left(\dfrac{az+b}{cz+d}\right) = \dfrac{v}{|cz+d|^2}$ and $\begin{pmatrix} \overrightarrow{x}_\gamma \\ \overrightarrow{y}_\gamma \end{pmatrix} = \gamma \begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}$ for $z = u + iv$ and $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Definition 3.1.2. A function F on L_0 which is left invariant under a finite-indexed subgroup Γ of $SL_2(\mathbb{Z}) \ltimes \mathbb{Z}^{2n}$ is said to be dominated by F_R if there is a constant $L > 0$ such that

$$\left| F\left((z,\phi); \begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}\right)\right| X_R(z) \le L + F_R\left(z; \begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}\right) \tag{3.10}$$

for all sufficiently large $R > 1$, uniformly for all $\left((z,\phi); \begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix}\right) \in L_0$, where

$$X_R(z) = \sum_{\gamma \in \pm\Gamma_\infty \setminus SL_2(\mathbb{Z})} \mathbb{1}_{[R,\infty)}(v_\gamma). \tag{3.11}$$

If F is a function on G^n invariant under Γ, a finite-indexed subgroup of $SL_2(\mathbb{Z})^n \ltimes \mathbb{Z}^{2n}$, we will say that F is a dominated function when there is a constant $L =$

$L(\overrightarrow{a}) > 0$ and a function $F_R = F_R(\overrightarrow{a})$ such that $F|_{L_{\overrightarrow{a}}} \circ \varphi_{\overrightarrow{a}}^{-1}$ satisfies the condition (3.10) for each positive integral vector $\overrightarrow{a}$.

We mention that if we let

$$F_{\overrightarrow{a}}(g) = F\left(gA_{\overrightarrow{a}}\right), \quad g \in G^n$$

for F defined as above, then $F_{\overrightarrow{a}}$ is also a Γ-invariant function on G^n so that $\overline{F_{\overrightarrow{a}}}$ and $\overline{F_{\overrightarrow{a}}|_{L_{\overrightarrow{a}}}}$ are well-defined.

Theorem 3.1.3. Let $F \geq 0$ be a continuous dominated function on G^n which is invariant under Γ, a finite-indexed subgroup of $SL_2(\mathbb{Z})^n \ltimes \mathbb{Z}^{2n}$ and $g_0 = \left(\left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array}\right); \left(\begin{array}{c} \overrightarrow{x} \\ 0 \end{array}\right)\right) \in G^n$, where $\overrightarrow{x}$ is a vector such that the components of the vector $({}^t\overrightarrow{x}, 1) \in \mathbb{R}^{n+1}$ are $\mathbb{Q}$-linearly independent. Let h be a piecewise continuous function on $\mathbb{R}/r\mathbb{Z}$, where $r = r\left(\Gamma_{\overrightarrow{a}}\right)$. Then,

$$\liminf_{v \to 0} \frac{1}{r} \int_0^r F\left(\left(a_j u + i a_j v, 0\right)_j; \left(\begin{array}{c} \overrightarrow{x} \\ \overrightarrow{0} \end{array}\right)\right) h(u)du \geq \int_{\Gamma_{\overrightarrow{a}} \backslash L_{\overrightarrow{a}}} \overline{F_{\overrightarrow{a}}|_{L_{\overrightarrow{a}}}} d\mu_{\overrightarrow{a}} \frac{1}{r} \int_0^r h(u)du \tag{3.12}$$

If we additionally assume that $\overrightarrow{x}$ is of diophantine type $\kappa < \frac{d-1}{d-2}$, then we have

$$\lim_{v \to 0} \frac{1}{r} \int_0^r F\left(\left(a_j u + i a_j v, 0\right)_j; \left(\begin{array}{c} \overrightarrow{x} \\ \overrightarrow{0} \end{array}\right)\right) h(u)du = \int_{\Gamma_{\overrightarrow{a}} \backslash L_{\overrightarrow{a}}} \overline{F_{\overrightarrow{a}}|_{L_{\overrightarrow{a}}}} d\mu_{\overrightarrow{a}} \frac{1}{r} \int_0^r h(u)du, \tag{3.13}$$

where $\mu_{\overrightarrow{a}}$ is the normalized Haar measure of $\Gamma_{\overrightarrow{a}} \setminus L_{\overrightarrow{a}}$.

Proof. Note that

$$\begin{aligned}
(a_j u + i a_j v, 0)_j &= \left(\begin{array}{cc} 1 & a_j u \\ 0 & 1 \end{array}\right)_j \left(\begin{array}{cc} (a_j v)^{1/2} & 0 \\ 0 & (a_j v)^{-1/2} \end{array}\right)_j \\
&= \left(\begin{array}{cc} 1 & a_j u \\ 0 & 1 \end{array}\right)_j \left(\begin{array}{cc} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{array}\right)_j \left(\begin{array}{cc} a_j^{1/2} & 0 \\ 0 & a_j^{-1/2} \end{array}\right)_j
\end{aligned}$$

so that

$$F\left((a_j u + i a_j v, 0)_j ; \begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix}\right) = F_{\vec{a}}\left((a_j u + i v, 0)_j ; \begin{pmatrix} \vec{x} \\ \vec{0} \end{pmatrix}\right)$$

whose domain is in $L_{\vec{a}}$ when u and v varies. The theorem comes from the theorem 5.1 [18].

□

Recall that

$$\Gamma^n = \left\{ \left(\begin{pmatrix} a_j & b_j \\ c_j & d_j \end{pmatrix} ; \begin{pmatrix} \frac{1}{2} a_j b_j \\ \frac{1}{2} c_j d_j \end{pmatrix} + \begin{pmatrix} m_j \\ n_j \end{pmatrix} \right)_j : \begin{array}{c} a_j, b_j, c_j, d_j, m_j, n_j \in \mathbb{Z} \\ a_j d_j - b_j c_j = 1 \end{array} \right\}.$$

It is true that Γ^n is of finite index in $SL_2(\mathbb{Z})^n \ltimes \left(\frac{1}{2}\mathbb{Z}\right)^{2n}$. So we remark that we can extend the above theorems to $SL_2(\mathbb{Z})^n \ltimes \left(\frac{1}{2}\mathbb{Z}\right)^{2n}$ rather than $SL_2(\mathbb{Z})^n \ltimes \mathbb{Z}^{2n}$.

Then the corresponding $\Gamma^n_{\vec{a}} = \Gamma^n \cap L_{\vec{a}} \subset L_{\vec{a}}$ is

$$\Gamma^n_{\vec{a}} = \left\{ \left(\begin{pmatrix} a & a_j b \\ \frac{1}{a_j} c & d \end{pmatrix} ; \begin{pmatrix} \frac{1}{2} a_j a b \\ \frac{1}{2} \frac{1}{a_j} c d \end{pmatrix} + \begin{pmatrix} m_j \\ n_j \end{pmatrix} \right)_j : \begin{array}{c} ad - bc = 1 \\ a, b, d, m_j, n_j \in \mathbb{Z} \\ c \in \mathrm{lcm}(a_j)\mathbb{Z} \end{array} \right\}$$

and $\varphi\left(\Gamma^n_{\vec{a}}\right) \subset L_0 = SL_2(\mathbb{R}) \ltimes \mathbb{R}^{2n}$ is

$$\varphi\left(\Gamma^n_{\vec{a}}\right) = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} ; \begin{pmatrix} \frac{1}{2} a_j a b \\ \frac{1}{2} c d \end{pmatrix} + \begin{pmatrix} m_j \\ n_j \end{pmatrix} \right)_j : \begin{array}{c} ad - bc = 1 \\ a, b, d, m_j, n_j \in \mathbb{Z} \\ c \in \mathrm{lcm}(a_j)\mathbb{Z} \end{array} \right\}.$$

Actually theorems 3.1.1 and 3.13 requires only that $\varphi\left(\Gamma^n_{\vec{a}}\right)$ is a lattice subgroup of L_0. We refer the next proposition for confirming this and for later calculation.

Proposition 3.1.4. The left action of the group $\varphi\left(\Gamma^n_{\vec{a}}\right)$ is properly discontinuous.

A fundamental domain of $\varphi\left(\Gamma_{\vec{a}}^{n}\right)$ in $\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2$ is given by

$$\mathfrak{F}_1 \times \left([0,1)^n \times \prod_{j=1}^{n} [0, a_j)\right),$$

where $\mathfrak{F}_1$ is a fundamental domain of $\varphi\left(\Gamma_{\vec{a}}^{n}\right)_1 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{array}{c} a, b, d \in \mathbb{Z} \\ c \in \mathrm{lcm}(a_j)\mathbb{Z} \end{array} \right\}$ in $\mathrm{SL}_2(\mathbb{R})$.

Proof. For any $\left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}; \begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix}\right)_j \in \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^{2n}$, by definition of $\mathfrak{F}_1$, there is a unique $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \varphi\left(\Gamma_{\vec{a}}^{n}\right)_1$ such that $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathfrak{F}_1$.

Consider the left action of

$$\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} \frac{1}{2}a_j ab \\ \frac{1}{2}cd \end{pmatrix} + \begin{pmatrix} m_j \\ a_j n_j \end{pmatrix}\right) \in \varphi\left(\Gamma_{\vec{a}}^{n}\right). \tag{3.14}$$

Then we get that

$$\begin{aligned} &\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}; \begin{pmatrix} \frac{1}{2}a_j ab \\ \frac{1}{2}cd \end{pmatrix} + \begin{pmatrix} m_j \\ a_j n_j \end{pmatrix}\right)_j \left(\begin{pmatrix} x & y \\ z & w \end{pmatrix}; \begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix}\right)_j \\ &= \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & y \\ z & w \end{pmatrix}; \begin{pmatrix} \frac{1}{2}a_j ab \\ \frac{1}{2}cd \end{pmatrix} + \begin{pmatrix} m_j \\ a_j n_j \end{pmatrix} + \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \xi_j \\ \eta_j \end{pmatrix}\right)_j. \end{aligned} \tag{3.15}$$

If we take

$$\begin{aligned} -m_j &= \left\lfloor \frac{1}{2}a_j ab + a\xi_j + b\eta_j \right\rfloor \\ -n_j &= \left\lfloor \frac{1}{a_j}\left(\frac{1}{2}cd + c\xi_j + d\eta_j\right) \right\rfloor, \end{aligned}$$

where $\lfloor \cdot \rfloor$ denotes a floor function, the element in (3.15) is contained in $\mathfrak{F}_1 \times \left([0,1)^n \times \prod_{j=1}^{n}[0, a_j)\right)$ and it is obvious that such an element of $\varphi\left(\Gamma_{\vec{a}}^{n}\right)$ in (3.14) is uniquely taken. $\square$

3.2 Proof of Theorem 1.0.4

Recall that for f and g in $\mathcal{S}(\mathbb{R}^n)$, we are interested in a function F on G^n defined by;

$$F\left(\left((z_j, \phi_j)\,;\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j\right) = \Theta_f\overline{\Theta_g}\left(\left((z_j;\phi_j)\,;\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j\right) \tag{3.16}$$

$$= \Theta_f\left(\left((z_j;\phi_j)\,;\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j\right)\overline{\Theta_g\left(\left((z_j;\phi_j)\,;\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j\right)},$$

which is invariant under the left multiplication of Γ^n. Although f and g are rapidly decreasing, $F = \Theta_f\overline{\Theta_g}$ is unbounded in general. Hence in order to apply Theorem 3.1.3, we need to check that F is dominated. For this, let us examine that for $\left((z,\phi)\,;\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j \in L_0 = \mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^{2n}$,

$$F_{\overrightarrow{a}} \circ \varphi^{-1}\left(\left((z,\phi)\,;\begin{pmatrix} x_j \\ y_j \end{pmatrix}\right)_j\right) \tag{3.17}$$

$$= \Theta_f\overline{\Theta_g}\left(\left(\left(\begin{pmatrix} a_j & 0 \\ 0 & a_j^{-1} \end{pmatrix}(z,\phi)\begin{pmatrix} a_j^{-1} & 0 \\ 0 & a_j \end{pmatrix};\begin{pmatrix} x_j \\ a_j^{-1}y_j \end{pmatrix}\right)\right)_j A_{\overrightarrow{a}}\right)$$

$$= \Theta_f\overline{\Theta_g}\left(\left(\begin{pmatrix} a_j & 0 \\ 0 & a_j^{-1} \end{pmatrix}(z,\phi);\begin{pmatrix} x_j \\ a_j^{-1}y_j \end{pmatrix}\right)\right)_j \tag{3.18}$$

$$
\begin{aligned}
&= (a_1 \cdots a_n)^{1/2} v^{n/2} \sum_{\overrightarrow{m}, \overrightarrow{n} \in \mathbb{Z}^n} f_\phi \left((m_1 - \frac{1}{a_1} y_1)\sqrt{a_1 v}, \ldots, (m_n - \frac{1}{a_n} y_n)\sqrt{a_n v} \right) \\
&\quad \times \overline{g_\phi \left((n_1 - \frac{1}{a_1} y_1)\sqrt{a_1 v}, \ldots, (n_n - \frac{1}{a_n} y_n)\sqrt{a_n v} \right)} \\
&\quad \times \exp(2\pi i (\overrightarrow{m} \cdot \overrightarrow{x} - \overrightarrow{n} \cdot \overrightarrow{x})) \cdot \exp\left(\pi i \sum_{j=1}^{n} a_j u \left((m_j - \frac{1}{a_j} y_j)^2 - (n_j - \frac{1}{a_j} y_j)^2 \right) \right),
\end{aligned}
$$

where $f_\phi = \left[\tilde{R}(i, \phi) f\right]$ and $g_\phi = \left[\tilde{R}(i, \phi) g\right]$.

Proposition 3.2.1. The function F in (3.16) is dominated function.

Proof. We modify the proof in [16], [17] and [18] for our situation. For each positive integral vector $\overrightarrow{a} = (a_1, \ldots, a_n)$, we set

$$
f^*(\overrightarrow{w}) = (a_1 \cdots a_n)^{1/2} \sup_{\phi \in \mathbb{R}} \left| f_\phi g_\phi \left(a_1^{-1/2} w_1, \ldots, a_n^{-1/2} w_n \right) \right|. \tag{3.19}
$$

We claim that f^* is also rapidly decreasing. Since f_ϕ and g_ϕ are in $\mathcal{S}(\mathbb{R}^n)$ and ϕ ranges over a compact set $[0, \pi]$, there exists a constant $C_R > 0$ for any $R > 1$ such that when $\|\overrightarrow{w}\| > R$,

$$
f^*(\overrightarrow{w}) \leq (a_1 \cdots a_n)^{1/2} \, C_R (1 + \left(\frac{1}{a_1} w_1^2 + \cdots + \frac{1}{a_n} w_n^2 \right)^{1/2})^{-2R}
$$

which shows that $f^* \in \mathcal{S}(\mathbb{R}^n)$. Let

$$
F_R \left(z; \begin{pmatrix} \overrightarrow{x} \\ \overrightarrow{y} \end{pmatrix} \right) = \sum_{\gamma \in \Gamma_\infty \backslash \mathrm{SL}_2(\mathbb{Z})} \sum_{\overrightarrow{h} \in \mathbb{Z}^n} f^* \left((\overrightarrow{y}_\gamma + \overrightarrow{h}) v_\gamma^{1/2} \right) v_\gamma^{n/2} \mathbb{1}_{[R, \infty)}(v_\gamma).
$$

By considering the tessellation of fundamental domains of $\mathrm{SL}_2(\mathbb{R})$ in $\mathbb{H}^2$, The set $\{\gamma.(u+iv) : \gamma \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } v_\gamma > 1\}$ is the same as the set $\{\gamma.(u+iv) : \gamma \in \Gamma_\infty \gamma_0\}$ for some $\gamma_0 \in \mathrm{SL}_2(\mathbb{Z})$ such that $v_{\gamma_0} > 1$. Hence if $v > R$ for sufficiently large $R > 1$, we have that

$$F_R\left(z;\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\right) = v^{n/2}\sum_{\vec{h}\in\mathbb{Z}^n}\left[f^*\left((\vec{h}+\vec{y})v^{1/2}\right)+f^*\left((\vec{h}-\vec{y})v^{1/2}\right)\right].$$

From (3.17),

$$\begin{aligned}
&F_{\vec{a}}\circ\phi^{-1}\left(\left((z,\phi);\begin{pmatrix}x_j\\ y_j\end{pmatrix}\right)_j\right)\\
&\quad=(a_1\cdots a_n)^{1/2}f_\phi\overline{g_\phi}\left((h_1-\frac{1}{a_1}y_1)\sqrt{a_1v},\ldots,(h_n-\frac{1}{a_n}y_n)\sqrt{a_nv}\right)+\mathcal{O}(v^{-R})
\end{aligned}$$

uniformly for all $\vec{y}\in\mathbb{R}^n$ such that $h_k-1/2\le h_k/a_k\le h_k+1/2$ for all $k=1,\ldots,n$. Thus by definition of f^*, it follows that

$$\begin{aligned}
\left|F_{\vec{a}}|_{L_{\vec{a}}}\circ\phi^{-1}\left((z,\phi);\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\right)\right| &\le f^*\left((a_1h_1-y_1)v^{1/2},\ldots,(a_nh_n-y_n)v^{1/2}\right)|+\mathcal{O}(v^{-R})\\
&\le F_R\left(z;\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\right)+\mathcal{O}(v^{-R})
\end{aligned}$$

since $(a_1h_1,\ldots,a_nh_n)\in\mathbb{Z}^n$. Hence we can find a constant $L>1$ such that

$$\left|F_{\vec{a}}|_{L_{0a}}\circ\varphi^{-1}\left((z,\phi);\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\right)\right|X_R(z)\le F_R\left(z;\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\right)+L$$

for sufficiently large $R>1$, uniformly for all $\left((z,\phi);\begin{pmatrix}\vec{x}\\ \vec{y}\end{pmatrix}\right)\in L_0$. $\square$

Proposition 3.2.2. Let F be a function defined in (3.16). Following the notation in the theorem 3.13 with $\mathrm{SL}_n(\mathbb{Z})\ltimes\left(\frac{1}{2}\mathbb{Z}\right)^{2n}$ instead of $\mathrm{SL}_n(\mathbb{Z})\ltimes\mathbb{Z}^{2n}$, we have that

$$\int_{\Gamma_{\vec{a}}\backslash L_{\vec{a}}}\overline{F_{\vec{a}}|_{L_{\vec{a}}}}d\mu_{\vec{a}}=\int_{\mathbb{R}^n}f(\vec{y})\overline{g(\vec{y})}d\vec{y}. \tag{3.20}$$

Proof. Recall that the map $f\mapsto f_\phi=\tilde{R}(i,\phi)f$ is a unitary operator for any $\phi\in$

$[0, \pi]$, that is, for $f, g \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f_\phi(\overrightarrow{y})\overline{g_\phi}(\overrightarrow{y})d\overrightarrow{y} = \int_{\mathbb{R}^n} f(\overrightarrow{y})\overline{g}(\overrightarrow{y})d\overrightarrow{y}. \tag{3.21}$$

As in the proof of the theorem 3.13 and the proposition 3.1.4

$$\begin{aligned}\int_{\Gamma^n_{\overrightarrow{a}}\backslash L_{\overrightarrow{a}}} \overline{F_{\overrightarrow{a}}|_{L_{\overrightarrow{a}}}}d\mu_{\overrightarrow{a}} &= \int_{\varphi(\Gamma^n_{\overrightarrow{a}})\backslash L_0} \overline{F_{\overrightarrow{a}}|_{L_{\overrightarrow{a}}} \circ \varphi^{-1}}d\mu_0 \\ &= \frac{1}{\prod_{j=1}^n a_j}\int_{\mathfrak{F}_1}\int_{[0,1)^n\times\prod_{j=1}^n[0,a_j)} \overline{F_{\overrightarrow{a}}|_{L_{\overrightarrow{a}}} \circ \varphi^{-1}}dgd\overrightarrow{x}d\overrightarrow{y},\end{aligned}$$

where dg is a normalized Haar measure of $\mathfrak{F}_1$ and $d\overrightarrow{x}d\overrightarrow{y}$ is a Lebesgue measure of $\mathbb{R}^{2n}$. By (3.17),

$$\begin{aligned}\int_{\Gamma^n_{\overrightarrow{a}}\backslash L_{\overrightarrow{a}}} \overline{F_{\overrightarrow{a}}|_{L_{\overrightarrow{a}}}}d\mu_{\overrightarrow{a}} = &\frac{1}{\left(\prod_{j=1}^n a_j\right)^{1/2}}\int_{\mathfrak{F}_1}\int_{[0,1)^n\times\prod_{j=1}^n[0,a_j)} v^{n/2} \\ &\times \sum_{\overrightarrow{m},\overrightarrow{n}\in\mathbb{Z}^n} f_\phi\left((m_1 - \frac{1}{a_1}y_1)\sqrt{a_1 v}, \dots, (m_n - \frac{1}{a_n}y_n)\sqrt{a_n v}\right) \\ &\times \overline{g_\phi\left((n_1 - \frac{1}{a_1}y_1)\sqrt{a_1 v}, \dots, (n_n - \frac{1}{a_n}y_n)\sqrt{a_n v}\right)} \exp(2\pi i(\overrightarrow{m}\cdot\overrightarrow{x} - \overrightarrow{n}\cdot\overrightarrow{x})) \\ &\times \exp\left(\pi i \sum_{j=1}^n a_j u\left((m_j - \frac{1}{a_j}y_j)^2 - (n_j - \frac{1}{a_j}y_j)^2\right)\right) dgd\overrightarrow{x}d\overrightarrow{y} \\ = &\frac{1}{\left(\prod_{j=1}^n a_j\right)^{1/2}}\int_{\mathfrak{F}_1}\int_{\prod_{j=1}^n[0,a_j)} v^{n/2} \times \sum_{\overrightarrow{m}\in\mathbb{Z}^n} f_\phi\left((m_1 - \frac{1}{a_1}y_1)\sqrt{a_1 v}, \dots, (m_n - \frac{1}{a_n}y_n)\sqrt{a_n v}\right) \\ &\times \overline{g_\phi\left((m_1 - \frac{1}{a_1}y_1)\sqrt{a_1 v}, \dots, (m_n - \frac{1}{a_n}y_n)\sqrt{a_n v}\right)}dgd\overrightarrow{y}.\end{aligned}$$

The last equality follows by integrating on $\overrightarrow{x}$ so that all factors in the summation are deleted except when $\overrightarrow{m} = \overrightarrow{n}$. For each v, we will reparametrize $\overrightarrow{y}$ to $\overrightarrow{y}' = \left(\sqrt{v/a_1}y_1, \dots, \sqrt{v/a_n}y_n\right)$.

This and the equality (3.21) show that

$$\int_{\Gamma^n_{\vec{a}}\backslash L_{\vec{a}}} \overline{F_{\vec{a}}|_{L_{\vec{a}}}} d\mu_{\vec{a}} = \frac{1}{\left(\prod_{j=1}^n a_j\right)^{1/2}} \int_{\mathfrak{F}_1}\int_{\mathbb{R}^n} v^{n/2} f_\phi\left(\sqrt{\frac{v}{a_1}}y_1,\dots,\sqrt{\frac{v}{a_n}}y_n\right)$$
$$\times\, \overline{g_\phi\left(\sqrt{\frac{v}{a_1}}y_1,\dots,\sqrt{\frac{v}{a_n}}y_n\right)} d\vec{y}\, dg$$
$$= \int_{\mathfrak{F}_1}\int_{\mathbb{R}^n} f_\phi(\vec{y}')\overline{g_\phi(\vec{y}')}d\vec{y}' dg = \int_{\mathbb{R}^n} f(\vec{y}')g(\vec{y}')d\vec{y}'.$$

□

We can take an integer $r = r\left(\Gamma^n_{\vec{a}}\right)$ in Theorem 3.1.3 of lattice subgroup

$$\Gamma^n_{\vec{a}} = \left\{ \left(\left(\begin{array}{cc} a & a_j b \\ \frac{1}{a_j}c & d \end{array}\right); \left(\begin{array}{c} \frac{1}{2}a_j ab \\ \frac{1}{2}\frac{1}{a_j}cd \end{array}\right) + \left(\begin{array}{c} m_j \\ n_j \end{array}\right)\right)_j : \begin{array}{c} ad-bc=1 \\ a,b,d,m_j,n_j \in \mathbb{Z} \\ c \in \mathrm{lcm}(a_j)\mathbb{Z} \end{array} \right\}$$

of $L_{\vec{a}}$ by 2 since $\Gamma^n_{\vec{a}}\Psi^2_{\vec{a}} = \Gamma^n_{\vec{a}}$ for any positive integral vector $\vec{a}$.

Corollary 3.2.3. Suppose that $f(\vec{w}) = \psi\left(\|\vec{w}\|^2\right)$ for a positive-valued function $\psi \in \mathcal{S}(\mathbb{R})$ and let $h : \mathbb{R}/2\mathbb{Z} \to \mathbb{R}_{\geq 0}$ and $\vec{x}$ be as in the theorem 3.13, then

$$\lim_{v\to 0}\frac{1}{2}\int_0^2 \left|\Theta_f\left((a_j u + i a_j v, 0)_j; \left(\begin{array}{c}\vec{x}\\ \vec{0}\end{array}\right)\right)\right|^2 h(u)du \tag{3.22}$$
$$= \mathrm{vol}(B^n)\frac{n}{2}\int_0^\infty \psi(r)^2 r^{n/2-1}dr\frac{1}{2}\int_0^2 h(u)du,$$

where $\mathrm{vol}(B^n)$ is the volume of a unit ball B^n in $\mathbb{R}^n$.

Now let us prove the theorem 1.0.4. As Section 2.6, let (f_k) and f'_k be non-decreasing and non-increasing sequences, respectively, such that $f_k \leq \mathbb{1}_{[0,1]} \leq f'_k$ and $|f_k - f'_k|_\infty \to 0$ as $k \to \infty$. By Section 2.6 and Corollary 3.2.3 with $h = 1$ and

$N = \lfloor 1/v \rfloor$, we obtain that

$$\begin{aligned}
\lim_{N\to\infty} \frac{1}{N^{n/2}} \sum_{d=1}^{N} \left| \left[r_{\vec{a}}(\vec{\alpha}) \right](d) \right|^2 &= (a_1 \cdots a_n)^{-1/2} \lim_{v\to 0} \lim_{k\to\infty} \frac{1}{2} \int_0^2 \left| \Theta_{f_k} \left((a_j u + i a_j v, 0)_j ; \begin{pmatrix} \vec{x} \\ \vec{y} \end{pmatrix} \right) \right|^2 du \\
&= (a_1 \cdots a_n)^{-1/2} \operatorname{vol}(B_n) \frac{n}{2} \int_0^\infty \mathbb{1}^2_{[0,1]}(r) r^{n/2-1} \, dr \\
&= \operatorname{vol}(\Omega_{\vec{a}}).
\end{aligned}$$

Part II

Quantitative Oppenheim Conjecture for S-arithmetic case

Chapter 4

Preliminaries

Let S_f be a finite set $\{p_1, \ldots, p_s\}$ of (distinct) odd prime numbers and $S = \{\infty\} \cup S_f$. Each element of S (S_f, respectively) will be called a place (finite place, respectively). For each $v \in S$, we fix a normalized valuation $|\cdot|_v$ on $\mathbb{Q}$; when $v = \infty$, $|\cdot|_v$ is the usual absolute value. We will simply denote $|\cdot|_v$ by $|\cdot|$ when the implication is clear. Let $\mathbb{Q}_v$ be the completion of $\mathbb{Q}$ with respect to the norm $|\cdot|_v$ and denote by $\mathbb{Q}_S$ the direct product of $\mathbb{Q}_v$ as v ranges over elements of S:

$$\mathbb{Q}_S = \prod_{v \in S} \mathbb{Q}_v = \mathbb{R} \times \prod_{p \in S_f} \mathbb{Q}_p.$$

We think of $\mathbb{Q}$ as the image under a diagonal embedding in $\mathbb{Q}_S$. Let us take

$$\begin{aligned} \mathbb{Z}_S &= \{x \in \mathbb{Q} : |x|_v \leq 1, \text{ for all } v \in S_f\} \\ &= \left\{ \frac{m}{p_1^{m_1} \cdots p_s^{m_s}} : m, m_1, \ldots, m_s \in \mathbb{Z} \right\} = \mathbb{Z}\left[\frac{1}{p_1}, \ldots, \frac{1}{p_s}\right] \subset \mathbb{Q}. \end{aligned}$$

The set $\mathbb{Z}_S$ is called the ring of S-adic integers.

4.1 Geometry of $\mathrm{SL}_n(\mathbb{Q}_S)/\mathrm{SL}_n(\mathbb{Z}_S)$

Recall that the quotient space $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ is identified with the space of unimodular lattices in $\mathbb{R}^n$. In the extension of the real case, we want to understand $\mathrm{SL}_n(\mathbb{Q}_S)/\mathrm{SL}_n(\mathbb{Z}_S)$ as the space of unimodular lattices in $\mathbb{Q}_S^n$. However, since $\mathbb{Q}_S$ is not a field, we need to modify the definition of lattices.

First observe that $\mathbb{Q}_S$ is a locally compact abelian group and $\mathbb{Z}_S$ is a discrete cocompact subgroup of $\mathbb{Q}_S$.

Proposition 4.1.1. A fundamental domain of $\mathbb{Z}_S$ in $\mathbb{Q}_S$ is given by $[0,1]\times\prod_{p\in S_f}\mathbb{Z}_p$.

Proof. Let us denote $x \sim x'$ when there is $z \in \mathbb{Z}_S$ such that $x = x' + z$. Assume that $S_f = \{p_1 < \cdots < p_s\}$.

For $x \in \mathbb{Q}_S$, denote $x = (x_0, x_1, \ldots, x_s)$, $x_0 \in \mathbb{R}$ and $x_i \in \mathbb{Q}_{p_i}$. We first show that there is $x_o \in [0,1] \times \prod_{p\in S_f}\mathbb{Z}_p$ such that $x \sim x_o$. If $x \in [0,1] \times \prod_{p\in S_f}\mathbb{Z}_p$, we are done. If not, let j be the largest integer such that $x_i \notin \mathbb{Z}_{p_i}$. Denote

$$x_s = a_{-m}p_j^{-m} + a_{-m+1}p_j^{-m+1} + \cdots + a_{-1}p_j^{-1} + a_0 + a_1 p_j + \cdots ,$$

where each $a_k \in \{0, 1, \ldots, p_j - 1\}$ and $m \geq 1$. We choose $x' = (x_0', x_1', \ldots, x_s') \sim x$ by subtracting $a_{-m}p_j^{-m} + a_{-m+1}p_j^{-m+1} + \cdots + a_{-1}p_j^{-1} \in \mathbb{Z}_S$ to x. We claim that $\max\{i : x_i' \notin \mathbb{Z}_{p_i}\} < j$. For this, it is enough to show that $x_i - a_{-m}p_j^{-m} + a_{-m+1}p_j^{-m+1} + \cdots + a_{-1}p_j^{-1} \in \mathbb{Z}_i$ for $i \geq j$ or equivalently, $a_{-m}p_j^{-m} + a_{-m+1}p_j^{-m+1} + \cdots + a_{-1}p_j^{-1} \in \mathbb{Z}_{p_i}$ for $i > j$. If $i > j$, since $0 \leq a_k \leq p_j - 1 < p_i - 1$ and p_j^{-k} is a unit for each $k \in \{-m, \ldots, -1\}$, $a_k p_j^k$ is in $\mathbb{Z}_{p_i}$ so that $a_{-m}p_j^{-m} + a_{-m+1}p_j^{-m+1} + \cdots + a_{-1}p_j^{-1}$ is also contained in $\mathbb{Z}_{p_i}$, which shows the claim. By induction, we can find $x'' = (x_0'', x_1'', \ldots, x_s'') \in \mathbb{Q}_S$ such that each x_i'' is in $\mathbb{Z}_{p_i}$. Finally by subtracting the maximal integer $\lfloor x_0'' \rfloor$ less than or equal to x_0'', we obtain $x_o \in [0,1]\times\prod_{p\in S_f}\mathbb{Z}_p$ with $x_o \sim x$.

The uniqueness follows from the fact that if $z \in \mathbb{Z}_S$, since $|z|_v \geq 1$ for some $v \in S$, $x + z \notin [0,1] \times \prod_{p\in S_f}\mathbb{Z}_p$ for any $x \in [0,1] \times \prod_{p\in S_f}\mathbb{Z}_p$. $\square$

Hence we can define an S-lattice in $\mathbb{Q}_S^n$ as follows;

Definition 4.1.2. A $\mathbb{Z}_S$-module Δ is an S-*lattice* in $\mathbb{Q}_S^n$ if there are $\overrightarrow{x}_1, \ldots, \overrightarrow{x}_n \in \mathbb{Q}_S^n$ such that

$$\Delta = \mathbb{Z}_S\overrightarrow{x}_1 \oplus \cdots \oplus \mathbb{Z}_S\overrightarrow{x}_n$$

and $\mathbb{Q}_S^n$ is generated by $\overrightarrow{x}_1, \ldots, \overrightarrow{x}_n$.

We say that an S-lattice Δ is *unimodular* if the product of absolute values of determinants of each matrix whose columns are $\overrightarrow{x}_1^v, \ldots, \overrightarrow{x}_n^v$, $v \in S$, is one. If we equip $\mathbb{Q}_S^n$ with the normalized measure invariant under translations(see Subsection 4.3.1), we see that unimodular S-lattices are those whose covolumes are one.

We can associate any $g \in \mathrm{SL}_n(\mathbb{Q}_S)$ with a unimodular S-lattice $g\mathbb{Z}_S^n$ and it is clear that $g_1\mathbb{Z}_S^n = g_2\mathbb{Z}_S^n$ if and only if $g_1^{-1}g_2 \in \mathrm{SL}_n(\mathbb{Z}_S)$. Hence we can take an injective map from $\mathrm{SL}_n(\mathbb{Q}_S)/\mathrm{SL}_n(\mathbb{Z}_S)$ into the space of unimodular S-lattices. It turns out that, unlike the real case, the space of unimodular S-lattices is bigger than $\mathrm{SL}_n(\mathbb{Q}_S)/\mathrm{SL}_n(\mathbb{Z}_S)$. To show this, for each $p \in S_f$, let

$$\mathrm{UL}_n(\mathbb{Q}_p) = \{g \in \mathbf{GL}_n(\mathbb{Q}_p) : |\det g| = 1\}.$$

In other words, $\mathrm{UL}_n(\mathbb{Q}_p)$ consists of elements whose determinants are units in $\mathbb{Z}_p$. Note that $\mathrm{SL}_n(\mathbb{Q}_p)$ is a p-adic Lie subgroup of $\mathrm{UL}_n(\mathbb{Q}_p)$ of positive codimension.

Proposition 4.1.3. The space of unimodular S-lattices can be identified with $\left(\mathrm{SL}_n(\mathbb{R}) \times \prod_{p \in S_f} \mathrm{UL}_n(\mathbb{Q}_p)\right)/\mathrm{SL}_n(\mathbb{Z}_S)$.

Proof. The map $g \mapsto g\mathbb{Z}_S^n$ is also well defined and injective so that it suffices to show the surjectivity. Let Δ be a given unimodular S-lattice and $\overrightarrow{x}_1, \ldots, \overrightarrow{x}_n$ be any $\mathbb{Z}_S$-basis of Δ. Denote $g_v = \left(\overrightarrow{x}_1^v, \ldots, \overrightarrow{x}_n^v\right) \in \mathbf{GL}_n(\mathbb{Q}_v)$ for each $v \in S$. By unimodularity,

$$(\det g_0, \det g_1, \ldots, \det g_s) = (\prod_{j=1}^{s} p_j^{-n_j}, p_1^{n_1}, \ldots, p_s^{n_s}).$$

If we replace $\overrightarrow{x}_1$ by $\overrightarrow{x}'_1 = \prod_{j=1}^{s} p_j^{n_j}\overrightarrow{x}_1$, then $\overrightarrow{x}'_1, \overrightarrow{x}_2, \ldots, \overrightarrow{x}_n$ are also an $\mathbb{Z}_S$-basis of Δ and the v-norm of the determinant of each $g'_v = (\overrightarrow{x}'^v_1, \overrightarrow{x}^v_2, \ldots, \overrightarrow{x}^v_n)$ are one. Take $g' = (g'_0, g'_1, \ldots, g'_s) \in \left(\mathrm{SL}_n(\mathbb{R}) \times \prod_{p \in S_f} \mathrm{UL}_n(\mathbb{Q}_p)\right)$. Then it is obvious that $g'\mathbb{Z}_S^n = \Delta$. $\square$

4.2 Quadratic forms in $\mathbb{Q}_S^n$ and orthogonal groups

Recall that a quadratic form Q in $\mathbb{Q}_S^n$ is a family of quadratic forms defined over each completion field $\mathbb{Q}_v$, $v \in S$. In $\mathbb{R}^n$, there are $n-1$ types of nondegenerate indefinite quadratic forms:

$$Q_o(\overrightarrow{x}) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_n^2, \quad 1 \leq p \leq n-1, q = n-p.$$

We say that a quadratic form Q have the signature (p, q) if there is $g \in \mathbf{GL}_n(\mathbb{R})$ such that $Q(\overrightarrow{x}) = Q_o(g\overrightarrow{x})$.

4.2.1 Quadratic forms over $\mathbb{Q}_p$

In this subsection, we review the classification of quadratic forms over $\mathbb{Q}_p$ for an odd prime p. It turns out that there are at most eight types of quadratic forms in $\mathbb{Q}_p^n$ up to $\mathbf{GL}_n(\mathbb{Q}_p)$. All notations and theorems follow Serre's book [25] and we will omit the reference.

Proposition 4.2.1. Let $\mathcal{U}_p = \mathbb{Z}_p - p\mathbb{Z}_p$ be the set of units in $\mathbb{Z}_p$, $(\mathcal{U}_p)_1$ a subset of $\mathcal{U}_p$ of the form $1 + p\mathbb{Z}_p$ and $\mathbb{F}_p = \{x \in \mathcal{U}_p : x^{p-1} = 1\}$. Then

$$\begin{aligned}\mathbb{Q}_p^* &= \{x \in \mathbb{Q}_p : x \text{ is invertible }\} \\ &\cong \{p^{-z} : z \in \mathbb{Z}\} \times (\mathcal{U}_p)_1 \times \mathbb{F}_p \\ &\cong \mathbb{Z} \times \mathbb{Z}_p \times \mathbb{Z}/(p-1)\mathbb{Z}.\end{aligned}$$

Corollary 4.2.2. Let $\mathbb{Q}_p^{*2} = \{x^2 : x \in \mathbb{Q}_p^*\}$. Then we have that $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2} = \{1, p, u_o, pu_o\}$, where $u_o \in \{2, 3, \ldots, p-1\}$ is not a square in the group $\mathbb{Z}/p\mathbb{Z}$.

In other words, there are four directions of square numbers in $\mathbb{Q}_p$ in contrast to the real field.

Definition 4.2.3 (Hilbert Symbol)**.** Let $a, b \in \mathbb{Q}_p^*$. The Hilbert symbol (a, b) is defined by:

$$(a,b) = \begin{cases} 1, & \text{if } z^2 - ax^2 - by^2 = 0 \text{ has a solution } (z,x,y) \neq (0,0,0) \text{ in } \mathbb{Q}_p^3; \\ -1, & \text{otherwise.} \end{cases}$$

Definition 4.2.4. We say that a (nondegenerate) quadratic form Q is *isotropic* if there is a nonzero vector $\overrightarrow{x} \in \mathbb{Q}_p^n$ such that $Q(\overrightarrow{x}) = 0$.

Proposition 4.2.5. Let Q be an isotropic quadratic form on $\mathbb{Q}_p^n$. Then there are two isotropic elements $\overrightarrow{x}, \overrightarrow{y} \in \mathbb{Q}_p^n$ such that $\overrightarrow{x}, \overrightarrow{y}$ form a hyperbolic plane, that is, Q is of the form

$$Q(x_1, \ldots, x_n) = x_1 x_n + Q(0, x_2, \ldots, x_{n-1}, 0)$$

with an appropriate basis of $\mathbb{Q}_p^n$.

Recall that a quadratic form Q is *indefinite* if it has values both less than 0 and greater than 0. For real, quadratic forms being isotropic is equivalent to being indefinite. However in the case of local fields, they are not equivalent.

Theorem 4.2.6. For any quadratic form Q on $\mathbb{Q}_p^n$, there exists an orthogonal basis with respect to Q, that is, Q can be expressed as

$$Q(x_1, \ldots, x_n) = a_1 x_1^2 + \cdots + a_n x_n^2$$

for some $a_1, \ldots, a_n \in \mathbb{Q}_p^n$.

Now we introduce two invariants of a quadratic form.

Definition 4.2.7. For a quadratic form Q on $\mathbb{Q}_p^n$,

(a) the *discriminant* is defined as

$$\mathrm{d}(Q) = a_1 \cdots a_n \in \mathbb{Q}_p^* / \mathbb{Q}_p^{*2},$$

where $Q \sim a_1 x_1^2 + \cdots + a_n x_n^2$;

(b) the *Hasse invariant* is given by

$$\varepsilon(Q) = \prod_{i<j} (a_i, a_j),$$

where (a_i, a_j) is the Hilbert symbol of $a_i, a_j \in \mathbb{Q}_p^*$.

Theorem 4.2.8. The discriminant $d(Q)$ and the Hasse invariant $\varepsilon(Q)$ are independent of the choice of basis of $\mathbb{Q}_p^n$.

Theorem 4.2.9. Two quadratic forms over $\mathbb{Q}_p$ are equivalent if and only if they have the same rank, same discriminant, and the same Hasse invariant.

The above theorem shows that in each dimension, there are at most eight different types of quadratic forms up to $\mathbf{GL}_n(\mathbb{Q}_p)$, $n = \mathrm{rank}(Q)$. Although a modification of the following proposition classifies arbitrary quadratic forms, we restrict our attention to isotropic quadratic forms of rank ≥ 3.

Proposition 4.2.10. A quadratic form Q is isotropic if and only if

1. $n = 2$ and $d = -1$ (in $\mathbb{Q}_p^*/\mathbb{Q}_p^{*2}$);
2. $n = 3$ and $(-1, -d) = \varepsilon$;
3. $n = 4$ and either $d \neq 1$ or $d = 1$ and $\varepsilon = (-1, -1)$;
4. $n \geq 5$.

Therefore there are 4 types of quadratic forms of rank 3, 7 types of quadratic forms of rank 4 and every quadratic forms of rank ≥ 5 are isotropic. We remark that values of the quadratic form of rank 4 defined by

$$Q(x_1, x_2, x_3, x_4) = x_1^2 + px_2^2 + ux_3^2 + pux_4^2$$

are spread all over $\mathbb{Q}_p$ however Q is not isotropic. Let us also mention that this quadratic form is the *reduced norm* of the unique non-commutative field of degree 4 over $\mathbb{Q}_p$, defined as

$$\{x_1 + ix_2 + jx_3 + kx_4 : x_i \in \mathbb{Q}_p\},$$

where the multiplication is determined by $i^2 = -p$, $j^2 = -u$, $ij = k = -ji$([25]).

Since we are interested in nondegenerate isotropic quadratic forms over $\mathbb{Q}_p$, let us summarize all such quadratic forms of rank ≥ 3.

Notations 4.2.11 (Standard Quadratic form)**.** For a prime p, let u_0 be a fixed element of $\mathbb{Z}_p$ such that $(u_o, p) = -1$. The standard (isotropic) quadratic form Q_o^p over $\mathbb{Q}_p^n$ of a given signature refer to one of followings:

1. $n = 3$
   - $x_1x_3 + \alpha x_2^2,\ \alpha \in \{1, u_o, p, pu_o\}$.
2. $n = 4$
   - $x_1x_4 + \alpha x_2^2 - x_3^2,\ \alpha \in \{1, u_o, p, pu_o\}$,
   - $\begin{cases} x_1x_4 + px_2^2 + x_3^2 & \text{if -1 is not a square in } \mathbb{Q}_p; \\ x_1x_4 + ux_2^2 + pux_2^2 & \text{if -1 is square in } \mathbb{Q}_p. \end{cases}$
   - $\begin{cases} x_1x_4 + pux_2^2 + x_3^2 \\ x_1x_4 + px_2^2 + ux_2^2 \end{cases}$
   - $x_1x_4 + px_2^2 + pux_3^2$
3. $n \geq 5$
   - $x_1x_n + \alpha x_2^2 - x_3^2 + x_4^2 + \cdots + x_{n-1}^2,\ \alpha \in \{1, u_o, p, pu_o\}$,
   - $\begin{cases} x_1x_n + px_2^2 + x_3^2 + x_4^2 + \cdots + x_{n-1}^2 \\ x_1x_4 + ux_2^2 + pux_2^2 + x_4^2 + \cdots + x_{n-1}^2 \end{cases}$
   - $\begin{cases} x_1x_n + pux_2^2 + x_3^2 + x_4^2 + \cdots + x_{n-1}^2 \\ x_1x_4 + px_2^2 + ux_2^2 + x_4^2 + \cdots + x_{n-1}^2 \end{cases}$
   - $x_1x_n + px_2^2 + pux_3^2 + x_4^2 + \cdots + x_{n-1}^2$
   - $\begin{cases} x_1x_n + px_2^2 + px_3^2 - x_4^2 + x_5^2 + \cdots + x_{n-1}^2 \\ x_1x_n - px_2^2 - ux_3^2 + pux_4^2 + x_5^2 + \cdots + x_{n-1}^2 \end{cases}$

From now on, we fix the determinant of bilinear map

$$B(\overrightarrow{x}, \overrightarrow{y}) = B^Q(\overrightarrow{x}, \overrightarrow{y}) = \frac{1}{2}\left(Q(\overrightarrow{x} + \overrightarrow{y}) - Q(\overrightarrow{x}) - Q(\overrightarrow{y})\right)$$

for any quadratic form Q.

4.2.2 Orthogonal groups

For a given quadratic form Q over $\mathbb{Q}_S$, since Q consists of each quadratic form over $\mathbb{Q}_\nu$, $\nu \in S$, the orthogonal group $\mathrm{SO}(Q)$ is of the form $\prod_{\nu\in S} \mathrm{SO}(Q^\nu)$, where

$$\mathrm{SO}(Q^\nu) = \left\{ g_\nu \in \mathrm{SL}_n(\mathbb{Q}_\nu) : Q(g_\nu \overrightarrow{x}_\nu) = Q(\overrightarrow{x}_\nu), \quad \overrightarrow{x} \in \mathbb{Q}_\nu^n \ \right\}.$$

We remark that there might be fewer types of orthogonal groups than those of quadratic forms in $\mathbb{Q}_p^n$. We give an example when $n = 3$ for later use. If we take $\lambda = d(Q)$, then the quadratic form λQ has the signature $(d(\lambda Q), \varepsilon(\lambda Q)) = (1, \varepsilon(Q))$. Since $\mathrm{SO}(\lambda Q) = \mathrm{SO}(Q)$ for any $\lambda \in \mathbb{Q}_p^*$, it follows that any orthogonal group of a quadratic form in $\mathbb{Q}_p^3$ is isomorphic to one of $\mathrm{SO}(x_1x_3 - x_2^2)$ or $\mathrm{SO}(x_1^2+px_2^2+px_3^2)$(if $(p,p) = -1$, $\mathrm{SO}(px_1^2+u_o x_2^2+pu_o x_3^2)$ if $(p,p) = 1$ respectively).

Let us introduce two subgroups of $\mathrm{SO}(Q_o)$ for the standard isotropic quadratic form Q_o with a given signature $\sigma = ((p,q), \prod_{p\in S_f}(d_p, \varepsilon_p))$ which will be play important roles in the proof of the main theorem. The first subgroup is commonly contained in $\mathrm{SO}(Q_o)$ for every signature σ defined as

$$\begin{aligned} A &= \{ \ a_t \in \mathrm{SO}(Q) : t = (t_0, t_1, \ldots, t_s) \in \mathbb{R} \times \mathbb{Z}^s \ \} \\ &= \left\{ \begin{array}{c} \mathrm{diag}(e^{-t_0}, 1, \ldots, 1, e^{t_0}) \times \prod_{i=1}^s \mathrm{diag}(p_i^{t_i}, 1, \ldots, 1, p_i^{-t_i}) \\ : t = (t_0, t_1, \ldots, t_s) \in \mathbb{R} \times \mathbb{Z}^s \end{array} \right\}. \end{aligned} \tag{4.1}$$

For the parameter $t = (t_0, t_1, \ldots, t_s)$, we will denote $T = (e^{t_0}, p_1^{-t_1}, \ldots, p_s^{-t_s})$ and $|T| = e^{t_0} \prod_{p\in S_f} p^{t_p}$.

Another significant subgroup of $\mathrm{SO}(Q_o)$ is a maximal compact subgroup. Recall that in the real case, a maximal compact subgroup K_0 of $\mathrm{SO}(Q_o^0)$ is unique up to an inner automorphism and we will think of K_0 as $\mathrm{SO}(n) \cap \mathrm{SO}(Q_o^0)$ which is isomorphic to $\mathrm{SO}(p) \times \mathrm{SO}(q)$.

In the p-adic case, it is known that there are possibly many (isomorphic classes of) maximal subgroups in p-adic linear groups. However, the subgroup $\mathrm{SL}_n(\mathbb{Z}_p) \subset \mathrm{SL}_n(\mathbb{Q}_p)$ plays a role of $\mathrm{SO}(n) \subset \mathrm{SL}_n(\mathbb{R})$ in the sense that the linear action of

$\mathrm{SL}_n(\mathbb{Z}_p)$ on $\mathbb{Q}_p^n$ preserves the p-adic norm of $\mathbb{Q}_p^n$ given by

$$\|\overrightarrow{x}_p\| = \max\{|x_i| : \overrightarrow{x}_p = (x_1, \dots, x_n)\}, \quad \overrightarrow{x}_p \in \mathbb{Q}_p^n.$$

We will check later that if the action $\wedge^i(\mathrm{SL}_n(\mathbb{Z}_p))$ on the i-th exterior power $\wedge^i(\mathbb{Q}_p^n)$ preserves the norm of $\wedge^i(\mathbb{Q}_p^n)$ for each $i = 1, \dots, n$.

It is a fact that the intersection $\mathrm{SL}_n(\mathbb{Z}_p) \cap \mathrm{SO}(Q_o^p)$ is a maximal compact subgroup of $\mathrm{SO}(\mathbb{Q}_p)$ and we denote it by K_p. Then a maximal compact subgroup K of $\mathrm{SO}(Q)$ is the product $\prod_{\nu \in S} K_\nu$.

As the orbit $\mathrm{SO}(p) \times \mathrm{SO}(q)$ of a nonzero vector $\overrightarrow{v}_0$ in $\mathbb{R}^n$, $n = p + q$, is the product of two spheres $r_1 S^{p-1} \times r_2 S^{q-1}$, where $r_1 = (v_1^2 + \cdots + v_p^2)^{1/2}$ and $r_2 = (v_{p+1}^2 + \cdots + v_n^2)^{1/2}$, we can figure out the orbit of K_ν of a nonzero vector $\overrightarrow{v}_p$ in $\mathbb{Q}_p^n$. We first recall Witt theorem.

Theorem 4.2.12 (Witt theorem)**.** Let (V, B) be a finite-dimensional vector space over an arbitrary field together with a nondegenerate symmetric or skew-symmetric bilinear form. If $f : U \to U'$ is an isometry between two subspaces of V then f extends to an isometry of V.

Proposition 4.2.13. Fix an arbitrary $\overrightarrow{w}_p \in \mathbb{Q}_p^n$. Then K_p acts on $\{\overrightarrow{v}_p \in \mathbb{Q}_p^n : Q(\overrightarrow{v}_p) = Q(\overrightarrow{w}_p)$ and $\|\overrightarrow{v}_p\|_p = \|\overrightarrow{w}_p\|_p\}$ transitively.

Proof. For simplicity, let us denote Q^p, K_p and $\overrightarrow{v}_p$ by Q, K and $\overrightarrow{v}$, etc. We need to prove that for given two vectors $\overrightarrow{v}_1$ and $\overrightarrow{v}_2$ in $\mathbb{Q}_p^n$ with the same p-norm and the same quadratic value, there is an element k in K such that $k.\overrightarrow{v}_1 = \overrightarrow{v}_2$. For convenience, we may assume that $\overrightarrow{v}_1 = \overrightarrow{e}_1$ and let $\overrightarrow{v}_2 = \overrightarrow{v} \in \mathbb{Z}_p^n$. Since the same argument will be applied to both isotropic or nonisotropic vectors, we will think of a quadratic form $Q(x_1, \dots, x_n)$ as one of

$$u_1 x_1^2 + \cdots + u_i x_i^2 + p(u_{i+1} x_{i+1}^2 + \cdots + u_n x_n^2) \quad \text{or}$$
$$x_1 x_2 + u_3 x_3^2 + \cdots + u_i x_i^2 + p(u_{i+1} x_{i+1}^2 + \cdots + u_n x_n^2).$$

depending on the case we want to treat. Here u_i's are units in $\mathbb{Z}_p$.

Then we can write the corresponding symmetric matrix $B = B_Q$ as $\begin{pmatrix} B' & \\ & pB'' \end{pmatrix}$, where B' and B'' are nondegenerate mod p.

The proposition demands to find a matrix $k \in K$ satisfying that

(a) $k.\overrightarrow{e}_1 = \overrightarrow{v}$ and

(b) ${}^tkBk = B$.

Since $\mathbb{Z}_p$ is the inverse limit of $\mathbb{Z}/p^j\mathbb{Z}$, $j \to \infty$, we will construct a chain $k^j \in \mathbf{GL}_n(\mathbb{Z}/p^{j+1}\mathbb{Z})$ such that

(a') $k^j.\overrightarrow{e}_1 = \overrightarrow{v}$ mod p^{j+1},

(b') ${}^tk^jBk^j = B$ mod p^{j+1} and

(c') $k^j = k^{j+1}$ mod p^{j+1}.

Then the inverse limit of $(k^j)_{j=0}^{\infty}$ will be an element satisfying the conditions (a) and (b). Let us denote $k^j = k_0 + pk_1 + p^2k_2 + \cdots + p^jk_j$.

Step1 $j = 0$. Let $k^0 = k_0 = \begin{pmatrix} X_0 & Y_0 \\ Z_0 & W_0 \end{pmatrix}$ depending on the size of B' and B''. By the condition (a'), the first column $\overrightarrow{v}_0$ of k_0 is given by $\overrightarrow{v}$ mod p. We want to find a solution of the following equation;

$$\begin{aligned}\begin{pmatrix} B' & \\ & 0 \end{pmatrix} &= \begin{pmatrix} {}^tX_0 & {}^tZ_0 \\ {}^tY_0 & {}^tW_0 \end{pmatrix}\begin{pmatrix} B' & \\ & 0 \end{pmatrix}\begin{pmatrix} X_0 & Y_0 \\ Z_0 & W_0 \end{pmatrix} \mod p \\ &= \begin{pmatrix} {}^tX_0B'X_0 & {}^tX_0B'Y_0 \\ {}^tY_0B'X_0 & {}^tY_0B'Y_0 \end{pmatrix} \mod p.\end{aligned}$$

By the assumption of our quadratic form, $Q(\mathrm{pr}_i(\overrightarrow{e}_1)) = Q(\mathrm{pr}_i(\overrightarrow{v}_0))$, where $\mathrm{pr}_i : (x_1, \ldots, x_n) \in (\mathbb{Z}/p\mathbb{Z})^n \to (x_1, \ldots, x_i) \in (\mathbb{Z}/p\mathbb{Z})^i$. Applying Theorem 4.2.12 to $\left((\mathbb{Z}/p\mathbb{Z})^i, B'\right)$, we can get an isometry X_0 satisfying that the first column is $\mathrm{pr}_i(\overrightarrow{v}_0)$ and ${}^tX_0B'X_0 = B'$ mod p. Since ${}^tX_0B'$ is invertible, we should take Y_0 as 0.Note that In this step, we can not determine Z_0 and W_0.

Step2 $j=1$. The matrix $k^1 = \begin{pmatrix} X_0+pX_1 & Y_0+pY_1 \\ Z_0+pZ_1 & W_0+pW_1 \end{pmatrix}$ has the first column as $\overrightarrow{v}_0 + p\overrightarrow{v}_1$ and should satisfy the following equations;

$$\begin{aligned} {}^tX_0B'X_0 + p({}^tX_0B'X_1 +{}^t X_1B'X_0 +{}^t Z_0B''Z_0) &= B' \bmod p^2 \\ {}^tX_0B'Y_0 + p({}^tX_1B'Y_0 +{}^t X_0B'Y_1 +{}^t Z_0B''W_0) &= 0 \bmod p^2 \\ {}^tY_0B'Y_0 + p({}^tY_0B'Y_1 +{}^t Y_1B'Y_0 +{}^t W_0B''W_0) &= pB'' \bmod p^2 \end{aligned}$$

Since $Y_0 = 0$ and ${}^tX_0B'X_0 = B'+pC_1^{11} \bmod p^2$ for some symmetric matrix C_1^{11}, we can reduce the above equations as;

$$\begin{aligned} C_1^{11} +^t X_0B'X_1 +^t X_1B'X_0 +^t Z_0B''Z_0 &= 0 \bmod p \\ {}^tX_0B'Y_1 +^t Z_0B''W_0 &= 0 \bmod p \\ {}^tW_0B''W_0 &= B'' \bmod p \end{aligned}$$

Take any Z_0 such that the first column is $(\nu_{i+1}, \ldots, \nu_n)$ and any W_0 satisfying ${}^tW_0B''W_0 = B'' \bmod p$ using Witt theorem. Then it suffices to show that there is a matrix X_1 with the given first column satisfying the equation $C_1 +^t X_0B'X_1 +^t X_1B'X_0 +^t Z_0B''Z_0 = 0 \bmod p$.

Step3 We claim that for a given invertible symmetric matrix A and any symmetric matrix C, there is a solution X of the equation ${}^tXA + AX = C$.

By considering the space of n by n matrices as a n^2- dimensional vector space, we can rewrite the above equation by;

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} [X]^1 \\ [X]^2 \\ \vdots \\ [X]^n \end{pmatrix} = \begin{pmatrix} [C]^1 \\ [C]^2 \\ \vdots \\ [C]^n \end{pmatrix}, \tag{4.2}$$

where a block matrix A_{ij} is defined by

$$A_{ij} = \begin{cases} \begin{pmatrix} a_{11} & a_{21} & \cdots & \cdots & a_{n1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 2a_{1i} & 2a_{2i} & \cdots & \cdots & 2a_{ni} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \cdots & \cdots & a_{nn} \end{pmatrix} & \text{when } A = (a_{st}), \quad i = j; \\ \text{All entries are zero except } j\text{-th row is} & \\ \qquad (a_{1i}, a_{2i}, \dots, a_{ni}), & i \neq j. \end{cases}$$

Since ${}^tXA + AX$ and C are both symmetric, after removing rows repeated(for example, one of second row and (n+1)th row corresponding to c_{12} and c_{21}) we get a linear equation from $(\mathbb{Z}/p\mathbb{Z})^{n^2}$ to $(\mathbb{Z}/p\mathbb{Z})^{n(n+1)/2}$. Furthermore, from the fact that A is invertible, the rank of this reduced linear equation is exactly $n(n+1)/2$ which tells us that there is a solution X.

Now let us show that $C_1 +^t X_0B'X_1 +^t X_1B'X_0 +^t Z_0B''Z_0 = 0 \bmod p$ has a solution when C_1, B', X_0, B'', Z_0 and the first column of X_1 are given. That is, the first n^2 by n submatrix ${}^t(A_{11}, \dots, A_{n1})$ of the equation (1) is erased together with the n variables $[X]^1$. Before that, let us assume that the removed row among the repeated rows in the above argument is always the former one. That is, in the example, we will remove the second row and leave $(n+1)$th row. Hence the entries of the $n(n+1)/2$ by n submatrix of the reduced matrix are zero except the first row, and consequently the rank of the linear equation $C_1 +^t X_0B'X_1 +^t X_1B'X_0 +^t Z_0B''Z_0 = 0 \bmod p$ is $n(n+1)/2-1$. On the other hand c_{11} in the equation (4.2) can be also removed since it is determined by A_{11} and $[X]^1$. Therefore if we check that the (1,1)-entry of $C_1 +^t X_0B'X_1 +^t X_1B'X_0 +^t Z_0B''Z_0 = 0 \bmod p$ holds, we can find a required matrix X_1. However this follows from the fact that $Q(\overrightarrow{e}_1) = Q(\overrightarrow{v}) = Q(\overrightarrow{v}_0 + p\overrightarrow{v}_1) \bmod p$.

Step4 In general, suppose that there exists a solution $k = k_0 + pk_1 + p^2k_2 +$

$\cdots + p^n k_n + \cdots$ satisfying (a) and (b). Then from the condition ${}^t kBk = B$,

$$\begin{aligned}
\begin{pmatrix} B' & \\ & pB'' \end{pmatrix} &= \sum_{j=0}^{\infty} p^j \left(\sum_{i=0}^{j} \begin{pmatrix} {}^tX_i & {}^tZ_i \\ {}^tY_i & {}^tW_i \end{pmatrix} \begin{pmatrix} B' & \\ & pB'' \end{pmatrix} \begin{pmatrix} X_{j-i} & Y_{j-i} \\ Z_{j-i} & W_{j-i} \end{pmatrix} \right) \\
&= \sum_{j=0}^{\infty} p^j \left(\sum_{i=0}^{j} \begin{pmatrix} {}^tX_iB'X_{j-i} & {}^tX_iB'Y_{j-i} \\ {}^tY_iB'X_{j-i} & {}^tY_iB'Y_{j-i} \end{pmatrix} \right) + \\
&\qquad \sum_{j=0}^{\infty} p^{j+1} \left(\sum_{i=0}^{j} \begin{pmatrix} {}^tZ_iB''Z_{j-i} & {}^tZ_iB''W_{j-i} \\ {}^tW_iB''Z_{j-i} & {}^tW_iB''W_{j-i} \end{pmatrix} \right).
\end{aligned}$$

Hence we should find X_j, Y_j, Z_{j-1} and W_{j-1} inductively. Take any Z_{j-1} with the first column ${}^t(v_{j-1}^{i+1}, \ldots, v_{j-1}^{n})$, where $\overrightarrow{v} = \overrightarrow{v}_0 + p\overrightarrow{v}_1 + \cdots + p^j \overrightarrow{v}_j + \cdots$. Then by step3 with the fact that $B(\overrightarrow{v}, \overrightarrow{v}) = \sum_{k=0}^{j} p^j \left(\sum_{i=0}^{k} B(\overrightarrow{v}_i, \overrightarrow{v}_{k-i}) \right) \bmod p^{j+1}$, we can find an appropriate X_j, W_{j-1} and Y_j satisfying the following equations.

$$\begin{aligned}
{}^tX_0B'X_j + {}^tX_jB'X_0 &= -\sum_{i=1}^{j-1} {}^tX_iB'X_{j-i} - \sum_{i=0}^{j-1} {}^tZ_iB''Z_{j-i-1} + C_j^{11} \mod p, \\
{}^tW_{j-1}B''W_0 + {}^tW_0B''W_{j-1} &= -\sum_{i=1}^{j-1} \left({}^tW_iB''W_{j-i-1} + {}^tY_iB'Y_{j-i} \right) + C_n^{22} \mod p, \\
{}^tX_0B'Y_j &= -\sum_{i=0}^{j-1} \left({}^tX_iB'Y_{j-i} + {}^tZ_iB''W_{j-i-1} \right) + C_j^{12} \mod p,
\end{aligned}$$

where C_j^{11}, C_j^{22} and C_j^{12} are obtained from the equations of the formal level $j-1$(See the step2).

Consequently, we can find $k \in \mathrm{GL}(n, \mathbb{Z}_p)$ such that ${}^tkBk = B$. Since $\det k = \pm 1$, k may not be an element of K. However we can easily find $k' \in K$ from k with $k'.\overrightarrow{e}_1 = \overrightarrow{v}$ using reflections in $\mathbb{Q}_p^n$. $\square$

4.3 Integration on $\mathbb{Q}_S$

We first recall a measure on a p-adic vector space $\mathbb{Q}_p^n$ which is an analogue of the canonical volume form $dx_1 \ldots dx_n$ in a real vector space $\mathbb{R}^n$ and define a measure on a S-arithmetic space $\mathbb{Q}_S^n$ as the product measure of measures on each $\mathbb{Q}_v^n$, $v \in S$. Moreover, we provide an appropriate measurement for parallelepipeds of $\mathbb{Z}_S$-(sub)lattices and orbits of maximal compact subgroups of orthogonal groups in $\mathbb{Q}_S^n$.

4.3.1 Measure on $\mathbb{Q}_p^n$

Let μ be the measure on $\mathbb{Q}_p$ which is invariant under translations, normalized by $\mu(\mathbb{Z}_p) = 1$. Then for a basic open set $a_p^{-b}\mathbb{Z}_p$ in $\mathbb{Q}_p$, we have

$$\mu(a + p^{-b}\mathbb{Z}_p) = p^b.$$

We can also integrate a real-valued function $f : \mathbb{Q}_p \to \mathbb{R}$ with respect to μ. For example, if we think of the p-norm $|\cdot| = |\cdot|_p$ as a function from $\mathbb{Q}_p$ to $\{p^z : z \in \mathbb{Z}\} \subset \mathbb{R}$, then

$$\begin{aligned}\int_{\mathbb{Z}_p} |x| d\mu(x) &= \sum_{z=0}^{\infty} \int_{p^z\mathbb{Z}_p - p^{z+1}\mathbb{Z}_p} |x| d\mu(x) = \sum_{z=0}^{\infty} p^{-z}(p^{-z} - p^{-z-1}) \\ &= (1 - \frac{1}{p})\frac{1}{1 - 1/p^2} = \frac{1}{1 + 1/p}.\end{aligned}$$

We can define μ also by using an inverse limit of a counting measure on $\mathbb{Z}/p^\ell\mathbb{Z}$, that is, for any compact set $Y \in \mathbb{Q}_p$, if $Y \subseteq p^{-r}\mathbb{Z}_p$ for some $r \in \mathbb{Z}$, then

$$\mu(Y) = \lim_{\ell\to\infty} \frac{\#Y_\ell}{\#\left(p^{-r}\mathbb{Z}_p/p^\ell\mathbb{Z}_p\right)} \cdot p^r = \lim_{\ell\to\infty} \frac{\#Y_\ell}{p^\ell},$$

where Y_ℓ is the image of Y under the projection $p^{-r}\mathbb{Z}_p \to p^{-r}\mathbb{Z}_p/p^\ell\mathbb{Z}_p \cong p^{-r}\mathbb{Z}/p^\ell\mathbb{Z}$.

We equip $\mathbb{Q}_p^n$ with the product measure μ^n and denote it by $dx_1 \cdots dx_n$ as in the real space $\mathbb{R}^n$. It is a fact that the change of variable formula also holds for the p-adic case([27] for example). In this subsection, we provide the proof for linear maps.

Proposition 4.3.1. For $g \in \mathbf{GL}_n(\mathbb{Q}_p)$, $g_*(dx_1 \cdots dx_n) = \det g \; dx_1 \cdots dx_n$.

Proof. Assume $n = 2$. Since $\mathbf{GL}_2(\mathbb{Q}_p)$ is generated by matrices of the forms

$$\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix}, \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$

where d_1, d_2 and c are in $\mathbb{Q}_p$, it suffices to show when g is one of the above matrices. For any basic open set $(a_1 + p^{-b_1}\mathbb{Z}_p) \times (a_2 + p^{-b_2}\mathbb{Z}_p) \subseteq \mathbb{Q}_p^2$,

$$\begin{aligned}
(1)\ \mu^2 &\left(\begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} \left((a_1 + p^{-b_1}\mathbb{Z}_p) \times (a_2 + p^{-b_2}\mathbb{Z}_p) \right) \right) \\
&= \mu^2 \left((a_1 d_1 + d_1 p^{-b_1}\mathbb{Z}_p) \times (a_2 d_2 + d_2 p^{-b_2}\mathbb{Z}_p) \right) \\
&= \mu \left(a_1 d_1 + d_1 p^{-b_1}\mathbb{Z}_p \right) \cdot \mu \left(a_2 d_2 + d_2 p^{-b_2}\mathbb{Z}_p \right) = |d_1 p^{-b_1}||d_2 p^{-b_2}| \\
&= |d_1 d_2| \mu^2 \left((a_1 + p^{-b_1}\mathbb{Z}_p) \times (a_2 + p^{-b_2}\mathbb{Z}_p) \right);
\end{aligned}$$

$$\begin{aligned}
(2)\ \mu^2 &\left(\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \left((a_1 + p^{-b_1}\mathbb{Z}_p) \times (a_2 + p^{-b_2}\mathbb{Z}_p) \right) \right) \\
&= \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p} \mathbb{1}_{\{(x, cx + y) : x \in a_1 + p^{-b_1}\mathbb{Z}_p, y \in a_2 + p^{-b_2}\mathbb{Z}_p\}} d\mu^2 \\
&= \int_{a_1 + p^{-b_1}\mathbb{Z}_p} \int_{\mathbb{Q}_p} \mathbb{1}_{(cx + a_2 + p^{-b_2}\mathbb{Z}_p)} dy dx = \int_{a_1 + p^{-b_1}\mathbb{Z}_p} |p^{-b_2}| dx \\
&= |p^{-b_1}||p^{-b_2}| = \mu^2 \left((a_1 + p^{-b_1}\mathbb{Z}_p) \times (a_2 + p^{-b_2}\mathbb{Z}_p) \right) \quad \text{and}
\end{aligned}$$

$$\begin{aligned}
(3)\ \mu^2 &\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \left((a_1 + p^{-b_1}\mathbb{Z}_p) \times (a_2 + p^{-b_2}\mathbb{Z}_p) \right) \right) \\
&= \mu^2 \left((a_2 d_2 + d_2 p^{-b_2}\mathbb{Z}_p) \times (a_1 d_1 + d_1 p^{-b_1}\mathbb{Z}_p) \right) \\
&= \mu \left(a_1 d_1 + d_1 p^{-b_1}\mathbb{Z}_p \right) \cdot \mu \left(a_2 d_2 + d_2 p^{-b_2}\mathbb{Z}_p \right) \\
&= \mu^2 \left((a_1 + p^{-b_1}\mathbb{Z}_p) \times (a_2 + p^{-b_2}\mathbb{Z}_p) \right).
\end{aligned}$$

The proof for general dimensions can be easily extended. ∎

We are now state the S-arithmetic Siegel integral formula describing the relation between the Haar measure of $\mathrm{SL}_n(\mathbb{Q}_S)/\mathrm{SL}_n(\mathbb{Z}_S)$ and the translation-invariant measure $d\overrightarrow{v}_1 \cdots d\overrightarrow{v}_n$ in $\mathbb{Q}_S^n$. Let f be a bounded function on $\mathbb{Q}_S^n - \{\overrightarrow{0}\}$ whose support is compact. For $g \in \mathrm{SL}_n(\mathbb{Q}_S)$, let

$$\tilde{f}(g) = \sum_{\overrightarrow{v} \in \mathbb{Z}_S^n} f(g\overrightarrow{v}). \tag{4.3}$$

More generally, $\tilde{f}$ is defined on the space of S-lattices; for a lattice Δ,

$$\tilde{f}(\Delta) = \sum_{\overrightarrow{v} \in \Delta} f(\overrightarrow{v}). \tag{4.4}$$

Lemma 4.3.2 ([15])**.** $\tilde{f}$ is integrable on $\mathrm{SL}_n(\mathbb{Q}_S)/\mathrm{SL}_n(\mathbb{Z}_S)$.

Then we can show the following.

Proposition 4.3.3. [S-arithmetic Siegel integral formula] Let f be a continuous function with compact support on $\mathbb{Q}_S^n - \{\overrightarrow{0}\}$, $G = \mathrm{SL}_n(\mathbb{Q}_S)$ and $\Gamma = \mathrm{SL}_n(\mathbb{Z}_S)$. Then

$$\int_{G/\Gamma} \widetilde{f} dg = \int_{\mathbb{Q}_S^n} f d\overrightarrow{v},$$

where dg denote the normalized Haar measure of G/Γ.

Proof. Since $\widetilde{f}$ is integrable, the map $f \mapsto \int_{G/\Gamma} \widetilde{f} dg$ defines a positive linear functional on $C_c(\mathbb{Q}_S^n)$, thus a Radon measure ν on $\mathbb{Q}_S^n$. Since ν is G-invariant, ν is also G-invariant hence ν is decomposed by a linear combination of measures supported on G-invariant components(G-orbits of vectors in $\mathbb{Q}_S^n$). In other words, ν is of the form $\sum c_I \otimes_{\nu \in S} \nu_{\mathbb{Q}_\nu}$, where $\nu_{\mathbb{Q}_\nu}$, $\nu \in S$, is either a Dirac detla measure $\delta_{\overrightarrow{0}_\nu}$ or the translation-invariant(Lebesgue for $\nu = \infty$) measure $d\overrightarrow{v}_\nu$ respectively.

Note that all coefficients, except $d\overrightarrow{v} = \prod_{\nu \in S} d\overrightarrow{v}_\nu$, are zero since an S-lattice Δ does not contain a vector $\overrightarrow{v}$ with $\overrightarrow{v}_\nu = 0$ for some $\nu \in S$. Hence it suffices to show that the coefficient of $d\overrightarrow{v}$ is one. Consider an increasing sequence of compactly supported continuous functions f_k approximating characteristic functions of $\{\overrightarrow{v} \in \mathbb{Q}_S^n : \|\overrightarrow{v}_\nu\| \leq (p_1 \cdots p_s)^k\}$, $k \in \mathbb{N}$. As $k \to \infty$, $\tilde{f}_k$ goes to the volume of the set $\{\overrightarrow{v} \in \mathbb{Q}_S^n : \|\overrightarrow{v}_\nu\| \leq (p_1 \cdots p_s)^k\}$. Hence if we take $f'_k = f_k/\operatorname{vol}(\{\overrightarrow{v} \in \mathbb{Q}_S^n : \|\overrightarrow{v}_\nu\| \leq$

$(p_1 \cdots p_s)^k\})$, then $(\tilde{f}'_k)$ is a sequence of uniformly bounded continuous functions so that

$$\lim_{k\to\infty} \int_{G/\Gamma} \tilde{f}'_k dg = 1 = \lim_{k\to\infty} \int_{\mathbb{Q}_S^n} f'_k d\overrightarrow{v},$$

which complete the proof of the proposition.

□

4.3.2 Norm of $\wedge^i(\mathbb{Q}_S^n)$

Recall that the inner product and its induced norm of the i-th exterior power $\wedge^i(\mathbb{R}^n)$ of $\mathbb{R}^n$ are defined by

$$< \overrightarrow{x}_1 \wedge \cdots \wedge \overrightarrow{x}_i, \overrightarrow{y}_1 \wedge \cdots \wedge \overrightarrow{y}_i >= \det\left(\overrightarrow{x}_k \cdot \overrightarrow{y}_j\right)_{1\le k,j\le i} \quad \text{and} \tag{4.5}$$

$$\|\overrightarrow{x}_1 \wedge \cdots \wedge \overrightarrow{x}_i\| = \left(< \overrightarrow{x}_1 \wedge \cdots \wedge \overrightarrow{x}_i, \overrightarrow{x}_1 \wedge \cdots \wedge \overrightarrow{x}_i >\right)^{1/2}. \tag{4.6}$$

for $\overrightarrow{x}_k, \overrightarrow{y}_j \in \mathbb{R}^n$, where $\cdot$ is the usual inner product of $\mathbb{R}^n$.

The norm (4.6) can be defined in a different way. Let $\{\overrightarrow{e}_1, \ldots, \overrightarrow{e}_n\}$ be the canonical basis of $\mathbb{R}^n$. One can easily check that the definition of (4.6) is equivalent to the following; for $\overrightarrow{x}_1 \wedge \cdots \wedge \overrightarrow{x}_i$, denote

$$\overrightarrow{x}_1 \wedge \cdots \wedge \overrightarrow{x}_i = \sum_J a_J \overrightarrow{e}_{j_1} \wedge \cdots \wedge \overrightarrow{e}_{j_i},$$

where $J = \{1 \le j_1 < \ldots < j_i \le n\}$ and $a_J \in \mathbb{R}$. Then we have that

$$\|\overrightarrow{x}_1 \wedge \cdots \wedge \overrightarrow{x}_i\| = \left(\sum_J a_J^2\right)^{1/2}.$$

Note that the group $O(n)$ of linear maps preserving the norm on $\mathbb{R}^n$ also preserves the norm on $\wedge^i(\mathbb{R}^n)$.

Now consider the i-th exterior power $\wedge^i(\mathbb{Q}_p^n)$. We also denote the canonical basis of $\mathbb{Q}_p^n$ by $\{\overrightarrow{e}_1, \dots, \overrightarrow{e}_n\}$. As above, let

$$\overrightarrow{x}_1 \wedge \cdots \wedge \overrightarrow{x}_i = \sum_J a_J \overrightarrow{e}_{j_1} \wedge \cdots \wedge \overrightarrow{e}_{j_i},$$

where $a_J \in \mathbb{Q}_p$. Then we can define the norm as follows:

$$\|\overrightarrow{x}_1 \wedge \cdots \wedge \overrightarrow{x}_i\| = \max_J \{a_J\}. \tag{4.7}$$

Proposition 4.3.4. The action $\wedge^i(\mathrm{SL}_n(\mathbb{Z}_p))$ preserves the norm (4.7) on $\wedge^i(\mathbb{Q}_p^n)$.

Proof. For any element k in $\mathrm{SL}_n(\mathbb{Z}_p)$,

$$\begin{aligned} \|kv\| &= \max\{\|\sum_I k_I a_I\| \text{ for some } k_I \in \mathbb{Z}_p\} \\ &\leq \max\{|a_I|_p\} \leq \|v\| \end{aligned}$$

and $\|kv\| \geq \|v\|$ if we consider $k^{-1} \in \mathrm{SL}_n(\mathbb{Z}_p)$.

□

4.3.3 Integration of submanifolds in $\mathbb{Q}_p^n$

Now let us introduce volume forms of submanifolds in $\mathbb{Q}_S^n$ inherited from the volume form $d\overrightarrow{v}$ of $\mathbb{Q}_S^n$, mainly in order to measure the volumes of orbits of maximal compact subgroups of orthogonal groups in $\mathbb{Q}_S^n$. Since such maximal compact subgroups are products of maximal compact subgroups at each place, in this subsection, it suffices to deal with volume forms of p-adic submanifolds in $\mathbb{Q}_p^n$.

We will provide two equivalent definitions. Recall that any orbit of the p-adic maximal compact subgroup $K_p = \mathrm{SL}_n(\mathbb{Z}_p) \cap \mathrm{SO}(Q^p)$ of $\mathrm{SO}(Q^p)$ for a given quadratic form Q^p on $\mathbb{Q}_p^n$ is of the form

$$\{\overrightarrow{w} \in \mathbb{Q}_p^n : \|\overrightarrow{w}\| = p^{c_1}, Q^p(\overrightarrow{w}) = c_2\},$$

where $c_1 \in \mathbb{Z}$ and $c_2 \in \mathbb{Q}_p$ are some constants. Note that the above K-orbit is an open subset of the $(n-1)$-dimensional p-adic variety in $\mathbb{Q}_p^n$ given by

$$\{\overrightarrow{w} \in \mathbb{Q}_p^n : Q^p(\overrightarrow{w}) = c_2\}.$$

Hence from now on, we will consider Y as a compact open subset of a d-dimensional p-adic variety in $\mathbb{Q}_p^n$ or a parallelepiped $\mathbb{Z}_p\overrightarrow{v}_1 \oplus \cdots \oplus \mathbb{Z}_p\overrightarrow{v}_d$ where $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_d \in \mathbb{Q}_p^n$. Then Y is contained in $p^{-r}\mathbb{Z}_p^n$ for some $r \in \mathbb{N}$.

Definition 4.3.5. Consider the projection $\pi_\ell : p^{-r}\mathbb{Z}_p^n \to p^{-r}\mathbb{Z}_p^n/p^\ell\mathbb{Z}_p^n$ and let $Y_\ell = \pi_\ell(Y)$. Then the volume form ν_d defined over Y is defined as follow;

$$\nu_d(Y) = \lim_{\ell\to\infty} \frac{\#Y_\ell}{p^{d\ell}} \tag{4.8}$$

As in the real case, we can define the volume form of a p-adic submanifold by describing the volume form of its tangent space.

Definition 4.3.6. Let Y be a parallelepiped $\mathbb{Z}_p\overrightarrow{v}_1\oplus\cdots\oplus\mathbb{Z}_p\overrightarrow{v}_d$, where $\overrightarrow{v}_1, \ldots, \overrightarrow{v}_d$ are linearly independent in $\mathbb{Q}_p^n$. Then the volume form ν'_d of the parallelepiped is defined as

$$\nu'_d(\mathbb{Z}_p\overrightarrow{v}_1 \oplus \cdots \oplus \mathbb{Z}_p\overrightarrow{v}_d) = \|\overrightarrow{v}_1 \wedge \cdots \wedge \overrightarrow{v}_d\|, \tag{4.9}$$

where $\|\cdot\|$ is the maximum norm of $\wedge^d(\mathbb{Q}_p^n)$ given in (4.7).

The following proposition says that the relationship between the counting measure ν'_d and ν has many similarities with that of the volume form of a submanifold in $\mathbb{R}^n$ and the Lebesugue measure of $\mathbb{R}^n$.

Proposition 4.3.7. [27, Theorem 9] Two measures ν_d and ν'_d are equivalent.

We remark that since the norm in (4.7) is invariant under $\mathrm{SL}_n(\mathbb{Z}_p)$, it implies that $\mathrm{SL}_n(\mathbb{Z}_p)$ is the set of elements in $\mathbf{GL}_n(\mathbb{Q}_p)$ whose linear actions on $\mathbb{Q}_p^n$ are rigid, which is the analogue of the action of $\mathrm{O}(n) \subset \mathbf{GL}_n(\mathbb{R})$ on $\mathbb{R}^n$. That is to say, we can think of columns of each element in $\mathrm{SL}_n(\mathbb{Q}_p)$ as an "orthonormal" basis in $\mathbb{Q}_p^n$.

Chapter 5

α-function

Recall that for a bounded function f on $\mathbb{Q}_S^n$ with compact support, $\tilde{f} : G/\Gamma \to \mathbb{R}$ defined as

$$\tilde{f}(g\Gamma) = \sum_{\overrightarrow{v} \in \mathbb{Z}_S^n} f(g\overrightarrow{v})$$

is an unbounded function in general. Since equidistribution theorems are usually stated for bounded functions, we need a modification to apply them for unbounded functions.

5.1 The rational subspace and the α-function

Let Δ be an S-lattice in $\mathbb{Q}_S^n$ and let $\overrightarrow{x}_1, \ldots, \overrightarrow{x}_n \in \mathbb{Q}_S^n$ be an $\mathbb{Z}_S$-basis of Δ;

$$\Delta = \mathbb{Z}_S\overrightarrow{x}_1 \oplus \cdots \oplus \mathbb{Z}_S\overrightarrow{x}_n.$$

We say that a subspace $L \subseteq \mathbb{Q}_S^n$ is Δ-*rational* if the intersection $L \cap \Delta$ is an S-lattice in L, that is, L is a subspace of $\mathbb{Q}_S^n$ generated by finite number of elements in Δ. Suppose that L is i-dimensional and $L \cap \Delta = \mathbb{Z}_S\overrightarrow{v}_1 \oplus \cdots \oplus \mathbb{Z}_S\overrightarrow{v}_i$. Define

$d_\Delta(L)$ or simply $d(L)$ as

$$\begin{aligned} d(L) = d_\Delta(L) &= \|\vec{v}_1 \wedge \cdots \wedge \vec{v}_i\|_S \\ &= \prod_{\nu \in S} \|(\vec{v}_1)_\nu \wedge \cdots \wedge (\vec{v}_i)_\nu\|_\nu, \end{aligned} \tag{5.1}$$

where $\|\cdot\|_\nu$ is defined in (4.6) for $\nu = \infty$ and (4.7) for $\nu \in S_f$. Note that $d(L)$ is the volume of the quotient space $L/(L \cap \Delta)$. If $L = \{\vec{0}\}$, we write $d(L) = 1$. Using properties of the norm in $\wedge^i(\mathbb{Q}_S^n)$, we can show the following lemma.

Lemma 5.1.1. Let Δ be an S-lattice in $\mathbb{Q}_S^n$ and let L and M be two Δ-rational subspaces. Then we have

$$d(L)d(M) \geq d(L \cap M)d(L + M). \tag{5.2}$$

Proof. We first examine that if L and M are Δ-rational, then $L \cap M$ and $L + M$ are also Δ-rational. Since a subspace of $\mathbb{Q}_S^n$ is Δ-rational if and only if it is generated by elements of Δ, $L + M$ is clearly a Δ-rational subspace of $\mathbb{Q}_S^n$. On the other hand, consider the projection map $\pi : \mathbb{Q}_S^n \to \mathbb{Q}_S^n/\Delta$. Note that the projection image $\pi(H)$ of a subspace H in $\mathbb{Q}_S^n$ is closed if and only if $H \cap \Delta$ is a lattice subgroup in H, that is, H is Δ-rational. Since π is proper, if we set $H = \pi^{-1}(\overline{\pi(L \cap M)}) = \pi^{-1}(\overline{\pi(L) \cap \pi(M)})$, we can easily check that H is Δ-rational and $H = L \cap M$.

Let $p : \mathbb{Q}_S^n \to \mathbb{Q}_S^n/(L \cap M)$. Since $d_\Delta(H) = d_{p(\Delta)}(p(H))d_\Delta(L \cap M)$ for any Δ-rational subspace H, the inequality (5.2) is equivalent to

$$d_{p(\Delta)}(L)d_{p(\Delta)}(M) \geq d_{p(\Delta)}(L + M).$$

Let $\{\vec{v}_1, \ldots, \vec{v}_\ell\}$ and $\{\vec{w}_1, \ldots, \vec{w}_m\}$ be bases of $p(L)$ and $p(M)$ consisting of elements in $p(\Delta)$ respectively. Since $(p(L) \cap p(\Delta)) + (p(M) \cap p(\Delta)) \subseteq p(L+M) \cap p(\Delta)$,

$$\begin{aligned} d_{p(\Delta)}(L)d_{p(\Delta)}(M) &= \|\vec{v}_1 \wedge \cdots \wedge \vec{v}_\ell\| \, \|\vec{w}_1 \wedge \cdots \wedge \vec{w}_m\| \\ &\geq \|\vec{v}_1 \wedge \cdots \wedge \vec{v}_\ell \wedge \vec{w}_1 \wedge \cdots \wedge \vec{w}_m\| \geq d_{p(\Delta)}(L + M). \end{aligned}$$

□

Now let us define the α-function.

Definition 5.1.2. For an S-lattice Δ, define

$$\alpha_i(\Delta) = \sup\left\{\frac{1}{d(L)} : L \text{ is a } \Delta\text{-rational subspace of dimension } i\right\},$$

where $1 \le i \le n$ and

$$\alpha(\Delta) = \max\{\alpha_i(\Delta) : 0 \le i \le n\}.$$

By definition, α_i- and α-functions are ranged over the space of unimodular S-lattices, however we prefer to think of them as functions on $\mathrm{SL}_n(\mathbb{Q}_S)/\mathrm{SL}_n(\mathbb{Z}_S)$. Since we have a good knowledge of the behavior of α_i- and α- functions on cusps in $\mathrm{SL}_n(\mathbb{Q}_S)/\mathrm{SL}_n(\mathbb{Z}_S)$(See Subsection 5.2, 5.3), the following S-adic Schmidt lemma plays an important role in modifying equidistribution theorems.

Lemma 5.1.3 (S-adic Schmidt Lemma)**.** Let f be a bounded function on $\mathbb{Q}_S^n$ with compact support. Then there exists a positive constant $c = c(f) > 0$ such that

$$\tilde{f}(\Delta) < c\alpha(\Delta) \tag{5.3}$$

for any S-lattice Δ in $\mathbb{Q}_S^n$ and the function $\tilde{f}$ defined in (4.4).

Proof. For any $\overrightarrow{v}, \overrightarrow{w} \in \mathbb{Q}_S^n$, since $\mathrm{Supp}(f)$ is compact and $\mathbb{Z}_S$ is discrete, there is $\ell(\overrightarrow{v}, \overrightarrow{w}) > 0$ such that

$$\|\overrightarrow{v}\| \# \left(\mathrm{Supp}(f) \cap (\overrightarrow{w} + \mathbb{Z}_S.\overrightarrow{v})\right) \le \ell(\overrightarrow{v}, \overrightarrow{w}). \tag{5.4}$$

Moreover, we can obtain that the set $\{(\overrightarrow{v}, \overrightarrow{w}) : \# \left(\mathrm{Supp}(f) \cap (\overrightarrow{w} + \mathbb{Z}_S.\overrightarrow{v})\right) > 0\}$ in $\mathbb{Q}_S^n \times \mathbb{Q}_S^n$ is also compact so that there is

$$\ell := \max_{\overrightarrow{v}, \overrightarrow{w} \in \mathbb{Q}_p^n} \ell(\overrightarrow{v}, \overrightarrow{w}) < \infty.$$

Roughly speaking, the quantity ℓ is an analogue of the diameter of $\mathrm{Supp}(f)$ in the real case. Note that ℓ depends only on the support of f.

For any S-lattice Δ, let $\{\overrightarrow{v}_1, \ldots, \overrightarrow{v}_n\}$ be a $\mathbb{Z}_S$-basis of Δ such that $\|\overrightarrow{v}_j\| < 1$ for $1 \le j \le i$ and $\|\overrightarrow{v}_j\| \ge 1$ for $i+1 \le j \le n$. Since the number of S-(sub)lattice points are inversely proportional to the volume of the (sub)lattice parallelepiped, by definitions of α_i, α and ℓ, it follows that

$$\begin{aligned}
\tilde{f}(\Delta) &\le \|f\|_\infty \# \left(\Delta \cap \mathrm{Supp}(f)\right) \\
&\le \|f\|_\infty \max_{\overrightarrow{w} \in \Delta} \# \left(\overrightarrow{w} + \mathbb{Z}_S \overrightarrow{v}_1 \oplus \cdots \oplus \mathbb{Z}_S \overrightarrow{v}_i \cap \mathrm{Supp}(f)\right) \\
&\qquad \cdot \max_{\overrightarrow{w} \in \Delta} \# \left(\overrightarrow{w} + \mathbb{Z}_S \overrightarrow{v}_{i+1} \oplus \cdots \oplus \mathbb{Z}_S \overrightarrow{v}_n \cap \mathrm{Supp}(f)\right) \\
&\le \|f\|_\infty \left(\ell^i \alpha_i(\Delta)\right) \cdot \ell^{n-i} \le \|f\|_\infty \ell^n \alpha(\Delta),
\end{aligned}$$

where $\|f\|_\infty$ denotes the supremum of values of f.

□

5.2 The limit of K-orbit in $\bigwedge^i(\mathbb{Q}_p^n)$

Since α_i-functions are determined by the norm of $\bigwedge^i(\mathbb{Q}_S^n)$, we are going to examine the limit of mean values of inverse of norms of a_tK-orbits of unit vectors in $\bigwedge^i(\mathbb{Q}_S^n)$ as $t \to \infty$(Proposition 5.2.4). Results of the real case can be found in Section 5 in [6].

For $p \in S_f$, denote $K_p = \mathrm{SL}_n(\mathbb{Z}_p) \cap \mathrm{SO}(Q^p)$ for the basic quadratic form Q^p with a given signature (d_p, ε_p)(Notations 4.2.11) and

$$A = \left\{a_t = \mathrm{diag}(p^t, 1, \ldots, 1, p^{-t}) \in \mathrm{SL}_n(\mathbb{Q}_p) : t \in \mathbb{Z}\right\}. \tag{5.5}$$

Theorem 5.2.1. [26, p-adic Implicit Function Theorem] Let X and Y be m- and n-dimensional p-adic manifolds, $x \in X, y \in Y$ and $\phi : X \to Y$ be a continuous function that is locally given by power series, such that $\phi(x) = y$. Then the followings are equivalent:

1. The rank of the linear map $d_x\phi : T_xX \to T_yY$ is r.

2. There exist coordinate systems (x_i) and (y_i) at x and y, respectively, such

that

$$\phi(x_1,\dots,x_m) = (x_1,\dots,x_r,\psi_{r+1}(x_1,\dots,x_m),\dots,\psi_n(x_1,\dots,x_m)),$$

where $\psi_{r+1},\dots,\psi_n$ are analytic functions.

Lemma 5.2.2. Let V be a finite dimensional vector space over $\mathbb{Q}_p$ and K be a compact subgroup of $\mathbf{GL}_n(V)$. Let W be a proper subspace of V such that for any finite index subgroup K' of K and any subspace W' of W, $K'W' \nsubseteq W'$. Then for any $v \in V$, the subset

$$\mathrm{tran}(v, W) = \{k \in \mathrm{K} : kv \in W\} \subseteq \mathrm{K} \tag{5.6}$$

has Haar measure zero.

Proof. Denote a Haar measure on K by m. Since K is compact, $m(\mathrm{K}) < \infty$. Suppose that $m(\mathrm{tran}(v, W))$ is positive. Take $W' \subseteq W$ of the smallest dimension such that $m(\mathrm{tran}(\overrightarrow{v}, W')) > 0$. For each $k \in \mathrm{K}$, set $kW' = \{k\overrightarrow{w} : w \in W'\}$. Clearly kW' is also a proper subspace and for each $k' \in \mathrm{K}$, we have $\mathrm{tran}(v, k'W') = k'\mathrm{tran}(v, W')$ and hence

$$m(\mathrm{tran}(v, k'W')) = m(\mathrm{tran}(v, W'))$$

for all $k' \in \mathrm{K}$. If $k'W' \neq W'$, since $k'W' \cap W'$ has dimension strictly lower than that of W', by minimality, the intersection

$$\mathrm{tran}(v, k'W') \cap \mathrm{tran}(v, W') \subseteq \mathrm{tran}(v, k'W' \cap W')$$

has measure zero. It implies that the subgroup $\{kW' : k \in \mathrm{K}\}$ if of finite index in K and the subspace W' is K'-invariant, which is a contradiction to the assumption of the Lemma. $\square$

Lemma 5.2.3. Let ρ be an analytic representation of $\mathrm{SL}_n(\mathbb{Q}_p)$ on a finite dimensional normed space V over $\mathbb{Q}_p$ such that any elements of $\rho\,(\mathrm{SL}_n(\mathbb{Z}_p))$ preserve the norm of V. For $g \in \mathrm{SL}_n(\mathbb{Q}_p)$ and $v \in V$, we will denote $\rho(g)(v)$ by gv. Let K be a closed connected and compact subgroup of $\mathrm{SL}_n(\mathbb{Z}_p)$ and let A be given in (5.5). Assume that V has a decomposition $V = W^- \oplus W^0 \oplus W^+$, where

$$\begin{aligned} W^- &= \left\{v \in V : a_t v = p^t v\right\}, \\ W^0 &= \{v \in V : a_t v = v\} \text{ and} \\ W^+ &= \left\{v \in V : a_t v = p^{-t} v\right\}. \end{aligned}$$

Let F be a closed subset of $\{v \in V : \|v\| = 1\}$. Suppose that the following conditions are satisfied:

1. The subspace $W^- + W^0$ satisfies the assumption of Lemma 5.2.2;
2. There is a positive integer ℓ such that for any nonzero $v \in W^-$,

$$\operatorname{codim}\left\{x \in \mathtt{Lie}(K) : xv \in W^-\right\} \geq \ell. \tag{5.7}$$

Then for any s, $0 < s < \ell$,

$$\lim_{t \to +\infty} \sup_{\vec{v} \in F} \int_K \frac{dm(k)}{\|a_t k v\|^s} = 0, \tag{5.8}$$

where m is the normalized Haar measure on K.

Proof. Let $p : V \to W^+ \oplus W^0$ and $p^+ : V \to W^+$ denote the natural orthogonal projections. For any $v \in V$ and $r \geq 0$, let us consider the following subsets of K:

$$\begin{aligned} &D(v, r) = \{k \in K : \|p(kv)\|_p \leq r\} \text{ and} \\ &D^+(v, r) = \left\{k \in K : \|p^+(kv)\|_p \leq r\right\}. \end{aligned}$$

We would like to show that the Haar measure of $D(v, r)$ is bounded by Cr^ℓ, where C is uniform over all $v \in F$.

Let us define the Lie algebra $\mathtt{Lie}(K)$ as the tangent space of the p-adic manifold K at point e. Consider the map $f_v : K \to W^+ \oplus W^0$ defined by $f_v(k) = p(kv)$. Note that since K acts by linear transformations, the derivative of f_v at the identity is given by

$$d_e f_v(x) = p(xv), \quad x \in \mathrm{Lie}(K) = T_e K. \tag{5.9}$$

For every $v \in V$, set

$$L_v = \{x \in \mathrm{Lie}(K) : xv \in W^-\} = \ker d_e f_v.$$

For $k \in K$, also define the map $m_k : K \to K$ by $m_k(k') = k'k$. Note that $m_k(e) = k$ and $d_e m_k : T_e K \to T_k K$ is an isomorphism. On the other hand, for $k, k' \in K$, we have

$$f_{kv}(k') = p(k'kv) = f_v(k'k) = (f_v \circ m_k)(k'),$$

implying that $f_{kv} = f_v \circ m_k$. Hence, by the chain rule

$$d_e f_{kv} = d_e(f_v \circ m_k) = d_k(f_v) \circ d_e m_k,$$

showing that

$$\mathrm{rank}\, d_e f_{kv} = \mathrm{rank}\, d_k f_v.$$

From the assumption, if $f_v(k) = 0$, then $kv \in W^-$, hence by the assumption of the theorem, $\mathrm{codim} \ker d_e f_{kv} \geq \ell$, which implies that $\mathrm{codim} \ker d_k f_v \geq \ell$.

We know that if $f : U \to V$ is an analytic function, then the map $x \mapsto \mathrm{rank}\, d_x f$ is lower semi-continuous:

$$\liminf_{y \to x} \mathrm{rank}\, d_y f \geq \mathrm{rank}\, d_x f.$$

This implies that for every $v \in W^-$, there exists an open subset U containing $[v] \in \mathbb{P}(V)$ such that for $[w] \in U$, we have $\mathrm{codim} \ker d_k f_w \geq \ell$. Since $\mathbb{P}(W^-)$ can be covered with finitely many of these open sets, there is $\epsilon > 0$ such that an ϵ-neighborhood of $\mathbb{P}(W^-)$ in $\mathbb{P}(V)$ over which the same holds, i.e. if $\|p(kv)\| \leq \epsilon \|v\|$, then the map f_v has a rank at least ℓ.

By the implicit function theorem, for any $k \in K$ for which $\|p(kv)\| \leq \epsilon \|v\|$, we can choose local coordinate systems for $\mathcal{U}_k \subset K$ at k and for $W^+ \oplus W^0$ in which

$f_v|_{\mathcal{U}_k}$ takes the form:

$$f_v(u_1, \dots, u_m) = (u_1, \dots, u_\ell, \psi_{\ell+1}, \dots, \psi_{m'}),$$

where $m = \dim K$, $m' = \dim W^0 + \dim W^+$ and $\psi_j(u_1, \dots, u_m)$ is an analytic function for $j = \ell + 1, \dots, m'$ depending smoothly on v.

If $r \le \epsilon$, for such a neighborhood $\mathcal{U}_k$, $k \in D(v, r)$, there is a nonnegative $\alpha = \alpha(k)$ satisfying that;

$$m\,(D(v, r) \cap \mathcal{U}_k) \sim \left(\frac{r}{\epsilon}\right)^{\ell+\alpha} m\,(D(v, \epsilon) \cap \mathcal{U}_k)$$

so that

$$\frac{m\,(D(v, r))}{r^\ell} \le \frac{1}{\epsilon^\ell} m\,(D(v, r))\,.$$

If $r \ge \epsilon$, then since $m(K) = 1$, it holds that

$$\frac{m\,(D(v, r))}{r^\ell} \le \frac{1}{\epsilon^\ell}.$$

By compactness of a ball $\{v \in V : \|v\| = 1\}$,

$$C := \sup_{\|v\|=1, r>0} \frac{m\,(D(v, r))}{r^\ell} < \infty. \tag{5.10}$$

We will now have to consider the set

$$D^+(v, 0) = \{k \in K : p^+(kv) = 0\} = \{k \in K : kv \in W^0 \oplus W^-\}.$$

By Lemma 5.2.2, we know that $m(D^+(v, 0)) = 0$. We claim that this and the compactness of F imply that

$$\lim_{r \to 0} \sup_{v \in F} m(D^+(v, r)) = 0. \tag{5.11}$$

Otherwise, pick a sequence $r_n \to 0$, and $v_n \in Q$ such that $m(D^+(v_n, r_n)) > \epsilon$ for some ϵ. Upon passing to a subsequence, we can assume that v_n converges to

v. Then

$$\epsilon \leq m(\{k : \|p^+(kv_n)\| \leq r_n\}) \subseteq m(\{k : \|p^+(kv)\| \leq r_n + \|v_n - v\|\}).$$

Since $r_n + \|v_n - v\| \to 0$, we obtain $m(D^+(v,0)) \geq \epsilon$, which is a contradiction.

Since $a_t|_{W^0} = \mathrm{Id}_{W^0}$ and (5.10), we see that for a positive integer t, $\|a_t kv\|^s \geq \|p(kv)\|^s$ hence

$$\int_{D(v,r)-D\left(v,\frac{r}{p}\right)} \frac{dm(k)}{\|a_t kv\|^s} \leq \frac{m\left(D(v,r)\right)}{(r/p)^s} \leq \left(p^{-s}C\right) r^{\ell-s},$$

for any $v \in V$, $\|v\| = 1$ and any $r > 0$. From the fact that $m\left(D(v,0)\right) = 0$,

$$\begin{aligned} \int_{D(v,r)} \frac{dm(k)}{\|a_t kv\|^s} &= \sum_{n=0}^{\infty} \int_{D(v,p^{-n}r)\setminus D(v,p^{-n-1}r)} \frac{dm(k)}{\|a_t kv\|^s} \\ &\leq \frac{C}{p^{-s} - p^{-\ell}} r^{\ell-s}. \end{aligned} \tag{5.12}$$

On the other hand, since $a_t|_{W^+} = p^{-t}\,\mathrm{Id}_{W^+}$, we get that $\|a_t kv\|^s \geq p^{ts}\|p^+(kv)\|^s$ so that for any $v \in V$, $r > 0$ and $r_1 > 0$,

$$\begin{aligned} \int_{K\setminus D(v,r)} \frac{dm(k)}{\|a_t kv\|^s} &\leq \int_{K-D^+(v,r_1)} \frac{dm(k)}{\|a_t kv\|^s} + \int_{D^=(v,r_1)\setminus D(v,r)} \frac{dm(k)}{\|a_t kv\|^s} \\ &\leq p^{-ts} r_1^{-s} + m\left(D^+(v,r_1)\right) r^{-s}. \end{aligned}$$

Therefore for a given $\epsilon_0 > 0$, since $\ell - s > 0$, take $r > 0$ and $r_1 > 0$ such that $Cr^{\ell-s}/\left(p^{-s} - p^{-\ell}\right) < \epsilon_0/3$ and $m\left(D^+(v,r_1)\right) r^{-s} < \epsilon_0/3$. Then we can take $t' > 0$ such that if $t > t'$, then $p^{-ts}r_1^{-s} < \epsilon_0/3$ so that

$$\int_K \frac{dm(k)}{\|a_t kv\|^s} < \epsilon_0.$$

$\square$

We now consider the i-th exterior power $\wedge^i(\mathbb{Q}_p^n)$ and let ρ_i be the i-th exterior representation of $\mathrm{SL}_n(\mathbb{Q}_p)$. Then V_i has a decomposition $V_i = W_i^- \oplus W_i^0 \oplus W_i^+$, where

$$\begin{aligned}
W_i^- &= \left\{v \in V_i : a_t v = p^t v\right\} \\
&= \left\langle\{\overrightarrow{e}_1 \wedge \overrightarrow{e}_{j_2} \wedge \cdots \wedge \overrightarrow{e}_{j_i} : 1 < j_2 < \cdots < j_i < n\}\right\rangle, \\
W_i^0 &= \{v \in V_i : a_t v = v\} \\
&= \left\langle\{\overrightarrow{e}_{j_1} \wedge \cdots \wedge \overrightarrow{e}_{j_i} : 1 < j_1 < \cdots < j_i < n\}\right\rangle \\
&\quad \oplus \left\langle\left\{\overrightarrow{e}_1 \wedge \overrightarrow{e}_{j_2} \wedge \cdots \wedge \overrightarrow{e}_{j_{i-1}} \wedge \overrightarrow{e}_n : 1 < j_2 < \cdots < j_{i-1} < n\right\}\right\rangle \text{ and} \\
W_i^+ &= \left\{v \in V_i : a_t v = p^{-t} v\right\} \\
&= \left\langle\left\{\overrightarrow{e}_{j_1} \wedge \cdots \wedge \overrightarrow{e}_{j_{i-1}} \wedge \overrightarrow{e}_n : 1 < j_1 < \cdots < j_{i-1} < n\right\}\right\rangle
\end{aligned}$$

with the standard basis $\{\overrightarrow{e}_1, \ldots, \overrightarrow{e}_n\}$ of $\mathbb{Q}_p^n$. Then we can show the following proposition.

Proposition 5.2.4. Let Q^p be the standard quadratic form on $\mathbb{Q}_p^n$, $n \geq 4$ with a given signature (d_p, ε_p) and A, V_i, W_i^-, W_i^0 and W_i^+ be as before. Let $K = K^p$ be the maximal compact subgroup $\mathrm{SL}_n(\mathbb{Z}_p) \cap \mathrm{SO}(Q^p)$. For each V_i, define

$$\mathsf{F}(i) = \left\{\overrightarrow{x}_1 \wedge \overrightarrow{x}_2 \wedge \cdots \wedge \overrightarrow{x}_i : \overrightarrow{x}_1, \overrightarrow{x}_2, \cdots, \overrightarrow{x}_i \in \mathbb{Q}_p^n\right\} \subset V_i \tag{5.13}$$

and $\mathsf{F}_i = \mathsf{F}(i) \cap \{v \in V_i : \|v\| = 1\}$. Then for any s, $0 < s < 2$,

$$\lim_{t\to\infty} \sup_{v \in \mathsf{F}_i} \int_K \frac{dm(k)}{\|a_t k v\|^s} = 0. \tag{5.14}$$

Proof. First observe the case when $Q^p(x_1, x_2, x_3, x_4) = x_1x_4 + \alpha_2 x_2^2 + \alpha_3 x_3^2$, where α_i is one of $\{\pm 1, \pm u_0, \pm p, \pm u_0 p\}$, $i = 2, 3$ and $u_0 \in \mathbb{Z}_p$ is such that $(u_0, p) = 1$.

Consider two subgroups of K;

$$U_2 = \left\{ \begin{pmatrix} 1 & & & \\ u_2 & 1 & & \\ & & 1 & \\ -\alpha_2 u_2^2 & -2\alpha_2 u_2 & & 1 \end{pmatrix} : u_2 \in \mathbb{Q}_p \right\} \quad \text{and} \tag{5.15}$$

$$U_3 = \left\{ \begin{pmatrix} 1 & & & \\ & 1 & & \\ u_3 & & 1 & \\ -\alpha_3 u_3^2 & & -2\alpha_3 u_3 & 1 \end{pmatrix} : u_3 \in \mathbb{Q}_p \right\}. \tag{5.16}$$

Then

$$\begin{aligned} \begin{pmatrix} 1 & & & \\ u_2 & 1 & & \\ & & 1 & \\ -\alpha_2 u_2^2 & -2\alpha_2 u_2 & & 1 \end{pmatrix} \overrightarrow{e}_1 \wedge \overrightarrow{e}_2 \wedge \overrightarrow{e}_3 &= \begin{pmatrix} 1 \\ u_2 \\ \\ -\alpha_2 u_2^2 \end{pmatrix} \wedge \begin{pmatrix} \\ 1 \\ \\ -2\alpha_2 u_2 \end{pmatrix} \wedge \begin{pmatrix} \\ \\ 1 \\ \\ \end{pmatrix} \\ &= \overrightarrow{e}_1 \wedge \overrightarrow{e}_2 \wedge \overrightarrow{e}_3 + 2\alpha_2 u_2 \ \overrightarrow{e}_1 \wedge \overrightarrow{e}_3 \wedge \overrightarrow{e}_4 + \alpha_2 u_2^2 \ \overrightarrow{e}_2 \wedge \overrightarrow{e}_3 \wedge \overrightarrow{e}_4, \end{aligned}$$

$$\begin{aligned} \begin{pmatrix} 1 & & & \\ & 1 & & \\ u_3 & & 1 & \\ -\alpha_3 u_3^2 & & -2\alpha_3 u_3 & 1 \end{pmatrix} \overrightarrow{e}_1 \wedge \overrightarrow{e}_2 \wedge \overrightarrow{e}_3 &= \begin{pmatrix} 1 \\ \\ u_3 \\ -\alpha_3 u_3^2 \end{pmatrix} \wedge \begin{pmatrix} \\ 1 \\ \\ \\ \end{pmatrix} \wedge \begin{pmatrix} \\ \\ 1 \\ -2\alpha_3 u_3 \end{pmatrix} \\ &= \overrightarrow{e}_1 \wedge \overrightarrow{e}_2 \wedge \overrightarrow{e}_3 - 2\alpha_3 u_3 \ \overrightarrow{e}_1 \wedge \overrightarrow{e}_2 \wedge \overrightarrow{e}_4 + \alpha_3 u_3^2 \ \overrightarrow{e}_2 \wedge \overrightarrow{e}_3 \wedge \overrightarrow{e}_4, \end{aligned}$$

$$\begin{aligned} &\begin{pmatrix} 1 & & & \\ u_2 & 1 & & \\ & & 1 & \\ -\alpha_2 u_2^2 & -2\alpha_2 u_2 & & 1 \end{pmatrix} (x \ \overrightarrow{e}_1 \wedge \overrightarrow{e}_2 + y \ \overrightarrow{e}_1 \wedge \overrightarrow{e}_3 + z \ \overrightarrow{e}_2 \wedge \overrightarrow{e}_3) \\ &\quad = x \ \overrightarrow{e}_1 \wedge \overrightarrow{e}_2 + 2\alpha_2 u_2 x \ \overrightarrow{e}_1 \wedge \overrightarrow{e}_4 - \alpha_2 u_2^2 x \ \overrightarrow{e}_2 \wedge \overrightarrow{e}_4 \\ &\qquad\qquad + y \ \overrightarrow{e}_1 \wedge \overrightarrow{e}_3 + (u_2 y + z) \ \overrightarrow{e}_2 \wedge \overrightarrow{e}_3 + \alpha_2(u_2^2 y + 2u_2 z) \overrightarrow{e}_3 \wedge \overrightarrow{e}_4, \end{aligned}$$

$$
\begin{pmatrix} 1 & & & \\ & 1 & & \\ u_3 & & 1 & \\ -\alpha_3 u_3^2 & & -2\alpha_3 u_3 & 1 \end{pmatrix} (x \overrightarrow{e}_1 \wedge \overrightarrow{e}_2 + y \overrightarrow{e}_1 \wedge \overrightarrow{e}_3 + z \overrightarrow{e}_2 \wedge \overrightarrow{e}_3)
$$

$$
\begin{aligned}
&= x \overrightarrow{e}_1 \wedge \overrightarrow{e}_2 - (u_3 x + z) \overrightarrow{e}_2 \wedge \overrightarrow{e}_3 + \alpha_3(u_3^2 x - 2u_3 z) \overrightarrow{e}_2 \wedge \overrightarrow{e}_4 \\
&\qquad + y \overrightarrow{e}_1 \wedge \overrightarrow{e}_3 - 2\alpha_3 u_3 y \overrightarrow{e}_1 \wedge \overrightarrow{e}_4 - \alpha_3 u_3^2 y \overrightarrow{e}_3 \wedge \overrightarrow{e}_4.
\end{aligned}
$$

Note that if K' is a finite index subgroup of K, then it is open in K and K' always intersects U_2 and U_3. The above computation shows that the subspace $W_i^- + W_i^0$ of V_i, for any i, satisfies the assumption of Lemma 5.2.2 with respect to K. Moreover, from the fact that subgroups U_2 and U_3 are defined near the identity, we obtain the condition (5.7) at least for $\ell = 2$ of Lemma 5.2.3.

By Lemma 5.2.3, for $0 < s < 2$, we can see that

$$
\lim_{t\to\infty} \sup_{v \in F_i} \int_K \frac{dm(k)}{\|a_t k v\|^s} = 0,
$$

for any i. Now let us show that (5.14) holds for general quadratic forms when $n \geq 4$.

Let Q^p be the standard quadratic form on $\mathbb{Q}_p^n$ of general dimension greater than 4. For any j_2, j_3 fixed, consider the embedding $\iota(j_2, j_3)$ from $SO(x_1x_4 + \alpha_{j_2} x_2^2 + \alpha_{j_3} x_3^3)$ to $SO(Q^p)$ induced by the following injection from $\mathbb{Q}_p^4$ into $\mathbb{Q}_p^n$;

$$
\begin{cases} \overrightarrow{e}_1 \mapsto \overrightarrow{e}_1, \\ \overrightarrow{e}_2 \mapsto \overrightarrow{e}_{j_2}, \\ \overrightarrow{e}_3 \mapsto \overrightarrow{e}_{j_3}, \\ \overrightarrow{e}_4 \mapsto \overrightarrow{e}_n. \end{cases} \tag{5.17}
$$

Clearly, the embedding $\iota(j_2, j_3)$ maps subgroups $U_2(\mathbb{Z}_p)$ and $U_3(\mathbb{Z}_p)$ into the maximal subgroup K of $SO(Q^p)$. Then for any $v \in W_i^- + W_i^0$, we can find an appropriate embedding $\iota(j_2, j_3)$ so that orbits $\iota(j_2, j_3)(U_2(\mathbb{Z}_p)).v$ and $\iota(j_2, j_3)(U_3(\mathbb{Z}_p)).v$ escape the subspace $W_i^- + W_i^0$ except the identity. Therefore by the same argument with the case $n = 4$, we can come to a conclusion (5.14).

$\square$

5.3 Behavior of the α-function

Recall that for a given quadratic form Q on $\mathbb{Q}_S^n$, A and K be subgroups of SO(Q) given by

$$
\begin{aligned}
\mathrm{A} &= \{\ a_t \in \mathrm{SO}(Q) : t = (t_0, t_1, \ldots, t_s) \in \mathbb{R} \times \mathbb{Z}^s\ \} \\
&= \left\{ \begin{array}{c} a_t = \mathrm{diag}(e^{-t_0}, 1, \ldots, 1, e^{t_0}) \times \prod_{i=1}^s \mathrm{diag}(p_i^{t_i}, 1, \ldots, 1, p_i^{-t_i}) \\ : t = (t_0, t_1, \ldots, t_s) \in \mathbb{R} \times \mathbb{Z}^s \end{array} \right\} \quad \text{and} \\
\mathrm{K} &= \mathrm{K}_0 \times \prod_{p \in S_f} K_p \\
&= (\mathrm{SO}(n) \cap \mathrm{SO}(Q^0)) \times \prod_{p \in S_f} (\mathrm{SL}_n(\mathbb{Z}_p) \cap \mathrm{SO}(Q^p)).
\end{aligned}
$$

We now study the behavior of the α-function in the direction a_t as $t \to \infty$. As mentioned before, $t \to \infty$ if and only if each t_i goes to infinity, $0 \le i \le s$. We also denote $t \succ t_o$ or $T = (e^{t_0}, p_1^{t_1}, \ldots, p_s^{t_s}) \succ T_o = (e^{(t_o)_0}, p_1^{(t_o)_1}, \ldots, p_s^{(t_o)_s})$ when $t_i > (t_o)_i$ for each $0 \le i \le s$.

Lemma 5.3.1. Let A and K be as above. Suppose that the signature of the real quadratic form is (p, q) with $p > 3$. Then for any s, $0 < s < 2$ and any $c > 0$, there exists $t \in \mathbb{R}_{>0} \times \mathbb{N}^s$ such that for any lattice Δ in $\mathbb{Q}_S^n$, it holds that

$$\int_K \alpha_i(a_t k \Delta)^s dm(k) < \frac{c}{2} \alpha_i(\Delta)^s + \omega^2 \max_{0<j<\min\{n-i,i\}} (\alpha_{i+j}(\Delta)\alpha_{i-j}(\Delta))^{s/2}, \tag{5.18}$$

where $\omega = |T| = e^{t_0} \times \prod_{p \in S_f} p^{t_p}$.

Proof. For given $c > 0$, by Proposition 5.2.4 one can find $t \in \mathbb{R}_{>0} \times \mathbb{N}^s$ such that for any $v \in F(i)$, where $F(i)$ is defined in (5.13) with $\|v\| = 1$,

$$\int_K \frac{dm(k)}{\|a_t k v\|^s} < \frac{c}{2}.$$

Then for any nonzero $v \in F(i)$,

$$\int_K \frac{dm(k)}{\|a_t k v\|^s} < \frac{c}{2} \frac{1}{\|v\|^s}. \tag{5.19}$$

For a given S-lattice Δ in $\mathbb{Q}_S^n$ and each i, there exists a Δ-rational subspace L_i of dimension i satisfying that

$$\frac{1}{d(L_i)} = \alpha_i(\Delta). \tag{5.20}$$

By substituting a wedge product of $\mathbb{Z}_S$-generators of $L_i \cap \Delta$ for v in (5.19), we have that

$$\int_K \frac{dm(k)}{d_{a_t k\Delta}(a_t k L_i)^s} < \frac{c}{2} \frac{1}{d_\Delta(L_i)^s}. \tag{5.21}$$

On the other hand, by definition of ω, we know that for $v \in F(i)$, $0 < i < n$,

$$\omega^{-1} \leq \frac{\|a_t v\|}{\|v\|} \leq \omega. \tag{5.22}$$

Define Ψ_i by the set of Δ-rational subspaces L of dimension i with $d_\Delta(L) < \omega^2 d_\Delta(L_i)$. We will show the inequality (5.18) by dividing cases when $\Psi_i = \{L_i\}$ and when $\Psi_i \neq \{L_i\}$. First assume that $\Psi_i = \{L_i\}$. Then by (5.22),

$$\begin{aligned} d_{a_t k\Delta}(a_t k L) &= \|a_t k v\| \geq \omega^{-1}\|v\| = \omega^{-1} d(L) \\ &\geq \omega d(L_i) = \omega\|v'\| \geq \|a_t k v'\| = d_{a_t k\Delta}(a_t k L_i), \end{aligned} \tag{5.23}$$

where v and v' are wedge products of $\mathbb{Z}_S$-generators of L and L_i respectively. By inequalities (5.21),(5.23) and the definition of α_i,

$$\int_K \alpha_i(a_t k\Delta)^s dm(k) < \frac{c}{2}\alpha_i(\Delta)^s. \tag{5.24}$$

Now we assume that $\Psi_i \neq \{L_i\}$. Let M be an element of Ψ_i different from L. Suppose that $\dim(M + L_i) = i + j$ for some $j > 0$. Using Lemma 5.1.1 and the inequality (5.22), we get that for any $k \in K$,

$$\alpha_i(a_t k \Delta) < \omega \alpha_i(\Delta) = \frac{\omega}{d(L_i)} < \frac{\omega^2}{(d(L_i)d(M))^{1/2}}$$
$$\leq \frac{\omega^2}{(d(L_i \cap M)d(L_i + M))^{1/2}} \leq \omega^2 \left(\alpha_{i+j}(\Delta)\alpha_{i-j}(\Delta)\right)^{1/2},$$

so that

$$\int_K \alpha_i(a_t k \Delta)^s dm(k) \leq \omega^2 \max_{0<j\leq \min\{n-i,i\}} \left(\alpha_{i+j}(\Delta)\alpha_{i-j}(\Delta)\right)^s. \tag{5.25}$$

Therefore we get the result (5.18) by combining (5.24) and (5.25).

□

Consider $\mathbf{GL}_n(\mathbb{Q}_p)$ for $p \in S_f$ and denote

$$D_n^+ = \left\{ \operatorname{diag}\left(p^{\lambda_1}, p^{\lambda_2}, \ldots, p^{\lambda_n}\right) \in \mathbf{GL}_n(\mathbb{Q}_p) : \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n, \lambda_i \in \mathbb{Z} \right\}.$$

For any $g \in \mathbf{GL}_n(\mathbb{Q}_p)$, consider the Cartan decomposition $g = k_1(g)d_2(g)k_2(g)$, $k_1(g), k_2(g) \in \mathbf{GL}_n(\mathbb{Z}_p)$ and $d(g) \in D_n^+$. We will denote $\lambda_i(g)$, $1 \leq i \leq n$ when $d(g) = \operatorname{diag}\left(p^{\lambda_1(g)}, p^{\lambda_2(g)}, \ldots, p^{\lambda_n(g)}\right)$.

Proposition 5.3.2. Let U be a neighborhood of e in $\mathrm{SL}_n(\mathbb{Z}_p)$ given by;

$$U = \{g \in \mathrm{SL}_n(\mathbb{Z}_p) : \|g - e\| < 1\}, \tag{5.26}$$

where $\|\cdot\|$ is the maximum p-norm of $\mathfrak{M}_n(\mathbb{Q}_p) \cong \mathbb{Q}_p^{n^2}$. Then for any $k \in U$ and $d, d' \in D_n^+$,

$$\lambda_i(dkd') = \lambda_i(d)\lambda_i(d') \text{ for } 1 \leq i \leq n. \tag{5.27}$$

Proof. We first note that $\|p^{\lambda_1(g)}\| = \|g\|$. Let us denote (i,j)-entry of g by g_{ij}. Let $d = \operatorname{diag}\left(p^{\lambda_1}, \ldots, p^{\lambda_n}\right)$ and $d' = \operatorname{diag}(p^{\lambda'_1}, \ldots, p^{\lambda'_n})$ be in D_n^+. Then $dkd' = (p^{\lambda_i + \lambda'_j} k_{ij})$. Since $|k_{ii}| = 1$ for any i and $|k_{ij}| < 1$ for any $i \neq j$, we get that

$$p^{\lambda_1(dkd')} = \frac{1}{\|dkd'\|} = \frac{1}{|p^{\lambda_1 + \lambda'_1} k_{11}|} = p^{\lambda_1 + \lambda'_1} = p^{\lambda_1(d)} p^{\lambda_1(d')}. \tag{5.28}$$

Consider the representation ρ_i of $\mathbf{GL}_n(\mathbb{Q}_p)$ on the i-th exterior product $\bigwedge^i(\mathbb{Q}_p^n)$ in the usual way and we denote the maximum p-norm of $\bigwedge^i(\mathbb{Q}_p^n)$ by $\|\cdot\|$, too. Note that

$$\|\rho_i(g)\| = \max\left\{ |\det(g_{JK})|_p : \begin{array}{c} J = \{1 \leq j_1 < \cdots < j_i \leq n\} \\ K = \{1 \leq k_1 < \cdots < k_i \leq n\} \end{array} \right\}, \tag{5.29}$$

where $g_{JK} = (g_{j_s k_t})_{1 \leq s,t \leq i}$. Since the element of $\mathbf{GL}_n(\mathbb{Z}_p)$ does not change the p-norm of $\bigwedge^i(\mathbb{Q}_p^n)$,

$$\|p^{\lambda_1(g)+\cdots+\lambda_i(g)}\| = \|\rho_i(g)\|.$$

If $k \in U$, by the assumption, since the p-norm of diagonals is 1,

$$|\det(k_{JK})| = \begin{cases} 1, & \text{if J=K;} \\ < 1, & \text{if J} \neq \text{K,} \end{cases} \tag{5.30}$$

so that $\|\rho_i(dkd')\| = 1/\left(p^{\sum_{j=1}^i \lambda_j + \sum_{j=1}^i \lambda'_j}\right)$. Hence we obtain that

$$p^{\lambda_1(dkd')+\cdots+\lambda_i(dkd')} = \frac{1}{\|\rho_i(dkd')\|} = p^{\sum_{j=1}^i \lambda_j + \sum_{j=1}^i \lambda'_j} = p^{\lambda_1(d)+\cdots\lambda_i(d)} p^{\lambda_1(d')+\cdots+\lambda_i(d')}. \tag{5.31}$$

□

We remark that the above proposition also holds for

$$D_-^n = \left\{ \operatorname{diag}\left(p^{\lambda_1}, p^{\lambda_2}, \ldots, p^{\lambda_n}\right) \in \mathbf{GL}_n(\mathbb{Q}_p) : \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n, \lambda_i \in \mathbb{Z} \right\},$$

instead of D_+^n.

Corollary 5.3.3. Let H be a simply connected simple algebraic group in $\mathbf{GL}_n(\mathbb{Q}_p)$ and K a maximal compact subgroup $\mathrm{SL}_n(\mathbb{Z}_p) \cap \mathrm{H}$ of H. For any $a_t = \operatorname{diag}(p^{-t}, 1, \ldots, 1, p^t)$ and $a_s = \operatorname{diag}(p^{-s}, 1, \ldots, 1, p^s)$, $s, t \geq 0$, there is a neighborhood U of e in K such that

$$a_t U a_s \subset \mathrm{K} a_t a_s \mathrm{K}.$$

Proof. Consider the Cartan decomposition of H(see Theorem 3.14 in [21]). Then for any $g \in H$, there exist $k_1(g)$ and $k_2(g)$ in K such that

$$a_t g a_s = k_1(g)\mathrm{diag}\left(p^{\lambda_1(a_t g a_s)}, \dots, p^{\lambda_n(a_t g a_s)}\right) k_2(g).$$

Take a neighborhood U of K satisfying the condition of Proposition 5.3.2. Then

$$\begin{aligned}\mathrm{diag}\left(p^{\lambda_1(a_t g a_s)}, \dots, p^{\lambda_n(a_t g a_s)}\right) &= \mathrm{diag}\left(p^{\lambda_1(a_t a_s)}, \dots, p^{\lambda_n(a_t a_s)}\right) \\ &= a_t a_s,\end{aligned}$$

hence we get the result.

□

Proposition 5.3.4. Let $H = SO(Q)$ and $K = O(n) \times \prod_{p \in S_f} SL_n(\mathbb{Z}_p) \cap H$. Let $\mathcal{F}$ be a family of strictly positive functions on H having the following properties:

(a) There exists a neighborhood $V(\lambda), \lambda > 1$ of the identity in H such that for any $f \in \mathcal{F}$,

- $\lambda^{-1} f(h) < f(gh) < \lambda f(h)$ for any $h \in H$ and $g \in V(\lambda)$ and
- $\mathrm{diag}\left(p^{-1}, 1, \dots, 1, p\right) \in V(\lambda)$ for each $p \in S_f$,
  where we consider $\mathrm{diag}\left(p^{-1}, 1, \dots, 1, p\right) \in SO(Q^p)$ as the element of $SO(Q)$;

(b) The functions $f \in \mathcal{F}$ are left K-invariant, that is, $f(Kh) = f(h)$, $h \in H$ and

(c) $\sup_{f \in \mathcal{F}} f(1) < \infty$.

Then there exists $0 < c = c(\mathcal{F}) < 1$ such that if for any $\mathbb{Q}$-linearly independent $t^0, \dots, t^s \succ 0$ and $b > 0$ there exists $B = B(t^0, \dots, t^s, b) < \infty$ with the following property: If $f \in \mathcal{F}$ and

$$\int_K f(a_{t^j} k h) dm(k) < c f(h) + b, \ \ 0 \le j \le s \tag{5.32}$$

for any $h \in H$, then

$$\int_K f(a_\tau k) dm(k) < B$$

for any $\tau > 0$.

Proof. For each $f \in \mathcal{F}$, if we consider

$$\tilde{f}(h) = \int_K f(hk)dm(k)$$

and $\tilde{\mathcal{F}} := \{\tilde{f} : f \in \mathcal{F}\}$, then properties (a), (b) and (c) also hold for $\tilde{\mathcal{F}}$. Hence we may assume that for any $f \in \mathcal{F}$ and $h \in H$,

$$f(KhK) = f(h) \tag{5.33}$$

and we need to show that

$$\sup_{\tau \succ 0} f(a_\tau) < B. \tag{5.34}$$

By corollary 5.3.3 and Lemma 5.11 in [6], we can take a neighborhood U of the identity in H such that $a_t U a_\tau \in KV(\lambda)a_t a_\tau K$ for any $t \succeq 0$ and $\tau \succeq 0$. By condition (a) and (5.33), it follows that

$$\int_K f(a_t k a_\tau)dm(k) \geq \int_{U\cap K} f(a_t k a_\tau)dm(k) > \frac{1}{\lambda} m(U \cap K) f(a_t a_\tau). \tag{5.35}$$

Suppose there are $t^0, \dots, t^s \succ 0$ and $b > 0$ satisfying that

$$\int_K f(a_{t^j} kh)dm(k) < \frac{1}{2\lambda} m(U \cap K) f(h) + b, \;\; 0 \leq j \leq s \tag{5.36}$$

for any $h \in H$. Then by taking $h = a_\tau$ in (5.36) and using (5.35), for any $\tau \succ 0$,

$$f(a_{t^j} a_\tau) < \frac{1}{2} f(a_\tau) + b', \;\; 0 \leq j \leq s, \tag{5.37}$$

where $b' = 2b/m(U \cap K)$. Let Θ be a semigroup generated by $t^0, \dots, t^s$. By the assumption, we can take a subset $P = \{(\tau_0, \tau_1, \dots, \tau_s) : 0 \leq \tau_j \leq r_j, 0 \leq j \leq s\}$ of $\mathbb{R} \times \mathbb{Z}^s$ such that $\{t \in \mathbb{R} \times \mathbb{Z}^s : t \succ 0\} \subseteq P\Theta$.

Since $\mathrm{diag}(e^{-\tau_0}, 1, \dots, 1, e^{\tau_0}), 0 \leq \tau_0 \leq r_0$ belongs to $V(\lambda)^i$ for some i, where $V(\lambda)^1 = V, V^i(\lambda) = V(\lambda)V(\lambda)^{i-1}$ and by the second assumption of the condition (a), there exists $k \in \mathbb{N}$ such that

$$\sup_{h\in H, \tau \in P} \frac{f(a_\tau h)}{f(h)} \le \lambda^k = b'' < \infty. \tag{5.38}$$

Now we claim that for any $\tau \succ 0$,

$$f(a_\tau) \le 2b'' \max\{f(1), b', 1\}. \tag{5.39}$$

For this, observe that for any $\tau \succ 0$, there exist $\eta \in P$ and $(\ell_0, \ell_1, \ldots, \ell_s) \in \mathbb{N}^{s+1}$ so that $\tau = \eta + (\ell_0 t^0 + \ell_1 t^1 + \ldots + \ell_s t^s)$. We will use the induction on $\ell = \sum_{j=0}^{s} \ell_s$. If $\ell = 0$, then it is done. If $\ell > 0$, there is $\ell_i > 0$ so that by (5.37) and (5.38),

$$\begin{aligned} f(a_\tau) &= f(a_\eta a_{(\ell_0 t^0 + \ell_1 t^1 + \ldots + \ell_s t^s)}) < b'' f(a_{(\ell_0 t^0 + \ell_1 t^1 + \ldots + \ell_s t^s)}) \\ &= b'' f(a_{t^i} a_{(\ell_0 t^0, \ldots, (\ell_i - 1) t^i, \ldots, \ell_s t^s)}) \\ &\le \frac{b''}{2} \left(f(a_{(\ell_0 t^0, \ldots, (\ell_i - 1) t^i, \ldots, \ell_s t^s)}) + b' \right) \\ &\le 2b'' \max\{f(1), b', 1\}. \end{aligned}$$

□

Theorem 5.3.5. Let K be given in Proposition 5.3.4. Assume that the signature of real quadratic form is (p, q) with $p \ge 3$. Then for any S-lattice Δ in $\mathbb{Q}_S^n$,

$$\sup_{t \succ 0} \int_K \alpha(a_t k \Delta)^s dm(k) < \infty. \tag{5.40}$$

The upper bound is uniform as Δ varies over compact sets in the space of S-lattices.

Proof. Define functions $f_0, f_1, \ldots, f_n$ on $H = SO(Q)$ by

$$f_i(h) = \alpha_i(h\Delta), h \in H.$$

Since α is bounded by the sum of f_i's, it suffices to show that

$$\sup_{t \succ 0, 0 \le i \le n} \int_K f_i^s(a_t k) dm(k) < \infty. \tag{5.41}$$

Let us check that $f_0, f_1, \ldots, f_n$ have the properties (a), (b) and (c).

Since we consider the action of H on $\wedge^i(\mathbb{Q}_S^n)$, $0 \le i \le n$, the fact that an element of K preserves the norm of $\wedge^i(\mathbb{Q}_S^n)$ shows that each f_i is left K invariant. For any $\lambda > 1$, we can take $V(\lambda)$ as the subset of H consisting of element whose nonzero minors are entirely laid in (λ^{-1}, λ). Moreover, if we set $\lambda = 2p_1 \cdots p_s$, where $S_f = \{p_1, \ldots, p_s\}$, each element $\mathrm{diag}(p_\nu^{-1}, 1, \ldots, 1, p_\nu)$ is included in $V(\lambda)$. Clearly

$$\sup_{0 \le i \le n} f_i(1) = \max \{ \text{ minors of } \Delta \} < \infty.$$

From Lemma 5.3.1 with $h\Delta$, for any i, $0 < i < n$ and $h \in H$, we can find $\mathbb{Q}$-linearly independent $t^0, \ldots, t^s \succ 0$ satisfying that

$$\int_K f_i^s(a_t k h) dm(k) < \frac{c}{2} f_i^2 + \omega^2 \max_{0<j<\min\{n-i,i\}} (f_{i+j} f_{i-j})^{s/2}, \ t = t_0, \ldots, t_s. \tag{5.42}$$

Multiplying any $0 < \epsilon < 1$ on both sides in (5.42) and after simple calculation, we get that

$$\begin{aligned} &\int_K \epsilon^{i(n-i)} f_i^s(a_t k h) dm(k) \\ &\qquad < \frac{c}{2} \epsilon^{i(n-i)} f_i^s + \epsilon \omega^2 \max_{0<j<\min\{n-i,i\}} \left(\epsilon^{(i+j)(n-i-j)} f_{i+j} \epsilon^{(i-j)(n-i+j)} f_{i-j} \right)^{s/2}. \end{aligned} \tag{5.43}$$

If we define

$$f_{\epsilon,s} = \sum_{0 \le i \le n} \epsilon^{i(n-i)} f_i^s,$$

since $\epsilon^{i(n-i)} f_i^s < f_{\epsilon,s}$, $f_0 = 1$ and $f_n = d(\lambda)^{-1}$, by putting $\epsilon = c/(2n\omega^2)$ in (5.43), we have the inequality (5.32) of Proposition 5.3.4.

$$\begin{aligned}\int_K f_{\epsilon,s}(a_t kh)dm(k) &< 1 + d(\Delta)^{-s} + \frac{c}{2} f_{\epsilon,s} + n\epsilon\omega^2 f_{\epsilon,s} \\ &= cf_{\epsilon,s} + 1 + d(\Delta)^{-1}. \end{aligned} \tag{5.44}$$

Let $\mathcal{C}$ be an arbitrary compact set of unimodular S-lattices Δ and consider $\mathcal{F}$ is a family of $f_{\epsilon,s}$ for each $\Delta \in \mathcal{C}$. Then $\mathcal{F}$ has properties of Proposition 5.3.4 with a neighborhood $V(\lambda) = \cap_{\Delta\in\mathcal{C}} V_\Delta(\lambda)$ of the identity in H for $\lambda = 2p_1 \cdots p_s$ and (5.44) tells that $f_{\epsilon,s} \in \mathcal{F}$ shares the constant c and b. Take c small enough so that the inequality (5.36) is satisfied. Since $\alpha_i(h\Delta)^s \leq \epsilon^{-i(n-i)} f_{\epsilon,s}(h)$, by Proposition 5.3.4, we conclude that there exists a constant $B > 0$ such that for each i, all $t \succ 0$ and all $\Delta \in \mathcal{C}$,

$$\int_K \alpha_i(a_t k\Delta)^s dm(k) < B.$$

□

Chapter 6

J_f-function and results

Recall that for a given quadratic form Q of signature $\sigma = \{(p,q)\} \times \prod_{p \in S_f}(d_p, \varepsilon_p)$, there is the standard quadratic form Q_o(See 4.2.11) such that $Q(\overrightarrow{x}) = Q_o(g\overrightarrow{x})$ for some $g \in G = \mathrm{SL}_n(\mathbb{Q}_S)$. Consider the orthogonal subgroup $H = \mathrm{SO}(Q_o)$ of Q_o and let K be the maximal compact subgroup $\left(\mathrm{SO}(n) \times \prod_{p \in S_f} \mathrm{SL}_n(\mathbb{Z}_p)\right) \cap H$.

6.1 p-adic analogue of the J_f function

In this section, we define a p-adic analogue of J_f function which is introduced at Section 3 in [6] for real. In the following two propositions, we denote Q_o^p, K_p and $a(t_p) \in H_p = \mathrm{SO}(Q_o^p)$ by Q_o, K and $a_t \in H$ respectively.

Let $\pi_1 : \mathbb{Q}_p^n \to \mathbb{Q}_p$ be the projection to the first coordinate and $\mathbb{Q}_{p,+}^n$ be the set $\{\overrightarrow{x} \in \mathbb{Q}_p^n : \pi_1(\overrightarrow{x}) \neq 0\}$

Proposition 6.1.1. Let f be a continuous function of compact support on $\mathbb{Q}_{p,+}^n$ which satisfies

$$f(ux_1, x_2, \ldots, x_{n-1}, u^{-1}x_n) = f(x_1, x_2, \ldots, x_n), \ \ \overrightarrow{x} \in \mathbb{Q}_{p,+}^n \tag{6.1}$$

for any unit $u \in \mathcal{U}_p = \mathbb{Z}_p - p\mathbb{Z}_p$ and ν be a non-negative continuous function on

$\mathbb{U}_p^n = \mathbb{Z}_p^n - p\mathbb{Z}_p^n$ such that

$$\nu(u\overrightarrow{x}) = \nu(\overrightarrow{x}), \;\; \overrightarrow{x} \in \mathbb{U}_p^n \tag{6.2}$$

for any unit $u \in \mathcal{U}_p$. Let J_f be a function on $\left(\{p^{-k} \in \mathbb{Q}_p : k \in \mathbb{Z}\} \cup \{0\}\right) \times \mathbb{Q}_p$ defined by

$$J_f(p^{-r}, \zeta) = \frac{1}{p^{r(n-2)}} \int_{\mathbb{Q}_p^{n-2}} f(p^{-r}, x_2, \ldots, x_{n-1}, x_n) dx_2 \cdots dx_{n-1}, \tag{6.3}$$

where in the integral, $x_n = p^r(\zeta - Q_o(0, x_2, \ldots, x_{n-1}, 0))$. Then for any $\epsilon > 0$, there is $t_p > 0$ such that for $t > t_p$ and every $\overrightarrow{v} \in \mathbb{Q}_p^n$ with $\|\overrightarrow{v}\| > p^{t_p}$, we have that

$$\left| c(K) p^{t(n-2)} \int_K f(a_t k \overrightarrow{v}) \nu(k^{-1}\overrightarrow{e}_1) dm(k) - J_f\left(\frac{p^t}{\|\overrightarrow{v}\|}, \zeta\right) \nu(\|\overrightarrow{v}\|\overrightarrow{v}) \right| < \epsilon, \tag{6.4}$$

where $c(K) = \mathrm{vol}(K.\overrightarrow{e}_1)/(1 - 1/p)$ and m is the normalized Haar measure on K.

Proof. Let π_i be the ith coordinate projection of $\mathbb{Q}_p^n$. By the assumption, the support of f is contained in

$$\mathrm{Supp} f \in \left(p^{-k}\mathbb{Z}_p \backslash p^{-k'}\mathbb{Z}_p\right) \times \prod_{i=2}^{n} p^{-k_i}\mathbb{Z}_p, \tag{6.5}$$

where $k, k', k_i \in \mathbb{Z}$ and $k > k'$. Since $a_p(t) = a_t = \mathrm{diag}\left(p^t, 1, \ldots, 1, p^{-t}\right)$, we see that if $f(a_t\overrightarrow{w}) \neq 0$ for $t + k' > \max\{k_2, \ldots, k_n\}$,

$$\|\overrightarrow{w}\| = |\pi_1(\overrightarrow{w})|. \tag{6.6}$$

Since the maximal compact subgroup $K = \mathrm{SL}(n, \mathbb{Z}_p) \cap \mathrm{SO}(Q_o)$ is norm-preserving, for any $\delta' > 0$, we can take $t' > 0$ such that if $f(a_t k\overrightarrow{v}) \neq 0$ and $t > t'$, $|\|\overrightarrow{v}\|(k\overrightarrow{v}) - u(\pi_1(k\overrightarrow{v}))\overrightarrow{e}_1| < \delta$ for a unit $u(\pi_1(k\overrightarrow{v})) \in \mathcal{U}$. Then,

$$\left| \|\overrightarrow{v}\|\overrightarrow{v} - k^{-1}(u(\pi_1(k\overrightarrow{v}))\overrightarrow{e}_1) \right| = \left| \|\overrightarrow{v}\|\overrightarrow{v} - u(\pi_1(k\overrightarrow{v}))(k^{-1}\overrightarrow{e}_1) \right| < \delta'. \tag{6.7}$$

Take $\delta' > 0$ small enough so that if $t > t'$, by (6.2), the following inequality holds.

$$\left|\nu(k^{-1}\overrightarrow{e}_1) - \nu(\|\overrightarrow{v}\|_p\overrightarrow{v})\right| < \epsilon. \tag{6.8}$$

On the other hand, we can take $t'' > 0$ such that for a given $\delta' > 0$, if $t > t''$,

$$\left\| a_t k\overrightarrow{v} - \left(\frac{1}{\|\overrightarrow{v}\|}p^t \cdot u(k\overrightarrow{v}), \pi_2(k\overrightarrow{v}), \ldots, \pi_{n-1}(k\overrightarrow{v}), \nu_n u(k\overrightarrow{v})^{-1}\right)\right\| < \delta_1,$$

where ν_n is determined by

$$Q_o\left(\frac{1}{\|\overrightarrow{v}\|}p^t \cdot u(k\overrightarrow{v}), \pi_2(k\overrightarrow{v}), \ldots, \pi_{n-1}(k\overrightarrow{v}), \nu_n u(k\overrightarrow{v})^{-1}\right) = Q_o(\overrightarrow{v})$$

$$\text{or } Q_o\left(\frac{1}{\|\overrightarrow{v}\|}p^t, \pi_2(k\overrightarrow{v}), \ldots, \pi_{n-1}(k\overrightarrow{v}), \nu_n\right) = Q_o(\overrightarrow{v}).$$

By (6.1) and taking $\delta'' > 0$ small enough, we can take $t'' > 0$ so that for $t > t''$,

$$\left| f(a_t k\overrightarrow{v}) - f\left(\frac{1}{\|\overrightarrow{v}\|}p^t, \pi_2(k\overrightarrow{v}), \ldots, \pi_{n-1}(k\overrightarrow{v}), \nu_n\right)\right| < \epsilon. \tag{6.9}$$

By considering the relation between the normalized Haar measure on K and a K-invariant measure of the orbit $K.\overrightarrow{v}$, together with estimation (6.7) and (6.9), we can approximate the integral in (6.3) by the integration on $K.(\|\overrightarrow{v}\|^{-1}\overrightarrow{e}_1) \subset \mathbb{Q}_p^n$; for $t > \max\{t', t''\}$,

$$\begin{aligned}\Bigg| \int_K & f(a_t k\overrightarrow{v})\nu(k^{-1}\overrightarrow{e}_1)dm(k) \qquad\qquad (6.10)\\ & - \int_{K.(\|\overrightarrow{v}\|^{-1}\overrightarrow{e}_1)} f\left(\frac{1}{\|\overrightarrow{v}\|}p^t, \pi_2(k\overrightarrow{v}), \ldots, \pi_{n-1}(k\overrightarrow{v}), \nu_n\right)\nu(\|\overrightarrow{v}\|\overrightarrow{v})d\mu\Bigg| \le \epsilon,\end{aligned}$$

where μ is the normalized Haar measure on $K.(\|\overrightarrow{v}\|^{-1}\overrightarrow{e}_1)$.

Recall that the orbit $K.\overrightarrow{v}$ is given by

$$K.\overrightarrow{v} = \left\{\overrightarrow{w} \in \mathbb{Q}_p^n : \|\overrightarrow{w}\| = \|\overrightarrow{v}\| \text{ and } Q_0(\overrightarrow{w}) = Q_0(\overrightarrow{v})\right\}.$$

Then $K.\overrightarrow{v} \cap \mathrm{Supp} f(a_t\cdot)$ is contained in the set

$$\left\{\overrightarrow{w} \in \mathbb{Q}_p^n : w_1 \in p^{-t_0}\mathbb{Z}_p \backslash p^{-t_0+1}\mathbb{Z}_p, w_i \in p^{-k_i}\mathbb{Z}_p, w_n \in p^{t+k_n}\mathbb{Z}_p\right\},$$

where $\|\overrightarrow{v}\| = p^{t_0}$. (Note that $k' < t_0 - t < k$.)

Take $t_p > 0$ large enough such that $t_p > \max\{t', t''\}$ and if $t > t_p$, the volume form of the orbit $K.(\|\overrightarrow{v}\|^{-1}\overrightarrow{e}_1)$ near $\mathrm{Supp} f(a_t\cdot)$ can be approximated by $dx_1 \cdots dx_{n-1}$(See Subsection 4.3.3). When $\overrightarrow{w} \in K.\overrightarrow{v} \cap \mathrm{Supp} f(a_t\cdot)$, using (6.1), we can show that

$$\begin{aligned} f(a_t k\overrightarrow{v}) &= f\left(w_1 p^t, w_2, \ldots, w_{n-1}, \frac{1}{w_1}\big(\zeta - Q_o(0, w_2, \ldots, w_{n-1}, 0)\big)p^{-t}\right) \\ &= f\left(\frac{1}{\|w_1\|}p^t u(w_1), w_2, \ldots, w_{n-1}, \frac{\|w_1\|}{u(w_1)}\big(\zeta - Q_o(0, w_2, \ldots, w_{n-1}, 0)\big)p^{-t}\right) \\ &= f\left(\frac{1}{\|w_1\|}p^t, w_2, \ldots, w_{n-1}, \|w_1\|\big(\zeta - Q_o(0, w_2, \ldots, w_{n-1}, 0)\big)p^{-t}\right), \end{aligned}$$

where $\zeta = Q_o(\overrightarrow{v})$ and $w_1 = \|w_1\|u(w_1)$. We see that $v_n = \|w_1\|\big(\zeta - Q_o(0, w_2, \ldots, w_{n-1}, 0)\big)p^{-t}$.

Hence (6.10) and the above argument shows that if $t > t_p$,

$$\begin{aligned} &\left|\int_K f(a_t k\overrightarrow{v})\nu(k^{-1}\overrightarrow{e}_1)dm(k) \right. &(6.11) \\ &\quad - \frac{1}{\mathrm{vol}\,(K.(\|\overrightarrow{v}\|^{-1}\overrightarrow{e}_1))}\int_{(p^{-t_0}\mathbb{Z}_p\backslash p^{-t_0+1}\mathbb{Z}_p)\times\mathbb{Q}_p^{n-2}} f\left(\frac{1}{\|\overrightarrow{v}\|_p}p^t, x_2, \ldots, x_{n-1}, v_n\right)\times \\ &\qquad\qquad \left. \nu(\|\overrightarrow{v}\|_p\overrightarrow{v})dx_1\cdots dx_{n-1}\right| < \epsilon \end{aligned}$$

Since $\mathrm{vol}(K.(\|\overrightarrow{v}\|^{-1}\overrightarrow{e}_1)) = (\|\overrightarrow{v}\|)^{n-1}\,\mathrm{vol}\,(K.\overrightarrow{e}_1) = p^{t_0(n-1)}\,\mathrm{vol}\,(K.\overrightarrow{e}_1)$ and by integrating on the first variable, the right integration in (6.11) is

$$\frac{1-1/p}{\mathrm{vol}(K.\overrightarrow{e}_1)}\cdot\frac{1}{p^{t_0(n-2)}}\int_{\mathbb{Q}_p^{n-2}} f\left(\frac{1}{\|\overrightarrow{v}\|_p}p^t, x_2, \ldots, x_{n-1}, v_n\right)\nu(\|\overrightarrow{v}\|_p\overrightarrow{v})dx_2\cdots dx_{n-1}.$$

Putting $p^{-r} = p^t/\|\overrightarrow{v}\|_p$ on $J_f(p^{-r}, \zeta)$, we get (6.4).

□

Proposition 6.1.2. [15] Suppose h is a real-valued continuous function of compact support on $\left(\mathbb{Q}_p^n - \{\overrightarrow{0}\}\right) \times \mathbb{Q}_p$. Then

$$\lim_{t\to\infty} \frac{1}{p^{(n-2)t}} \int_{\mathbb{Q}_p^n} h(p^t \overrightarrow{v}, Q_o(\overrightarrow{v}))d\overrightarrow{v}$$
$$= \mathrm{vol}(K.\overrightarrow{e}_1) \sum_{z=-\infty}^{\infty} p^{(n-2)z} \int_K \int_{\mathbb{Q}_p} h(p^{-z}k\overrightarrow{e}_1, \zeta)d\zeta dm(k), \quad (6.12)$$

where vol $=$ vol$_{n-1}$ denote the $(n-1)$-dimensional volume in $\mathbb{Q}_p^n$.

Proof. Note that $\mathbb{Q}_p^n = \sum_{z=-\infty}^{\infty} p^{-z}\mathbb{U}_p^n = \sum_{z=-\infty}^{\infty} \left(p^{-z}\mathbb{Z}_p^n - p^{-z+1}\mathbb{Z}_p^n\right)$. By change of variables,

$$\begin{aligned}
&\lim_{t\to\infty} \frac{1}{p^{t(n-2)}} \int_{\mathbb{Q}_p^n} h\left(p^t\overrightarrow{v}, Q_o(\overrightarrow{v})\right) d\overrightarrow{v} \\
&= \lim_{t\to\infty} p^{2t} \int_{\mathbb{Q}_p^n} h\left(\overrightarrow{v}, p^{-2t}Q_o(\overrightarrow{v})\right) d\overrightarrow{v} \\
&= \lim_{t\to\infty} \sum_{z=-\infty}^{\infty} p^{2t} \int_{p^{-z}\mathbb{U}_p^n} h\left(\overrightarrow{v}, p^{-2T}Q_o(\overrightarrow{v})\right) d\overrightarrow{v} \\
&= \lim_{t\to\infty} \sum_{z=-\infty}^{\infty} p^{2t} \int_{\mathbb{Q}_p} \mathbb{1}_{\left(p^{-z}\mathbb{U}_p^n\right)} h\left(\overrightarrow{v}, p^{-2t}Q_o(\overrightarrow{v})\right) d\overrightarrow{v} \\
&= \sum_{r=-\infty}^{\infty} \lim_{t\to\infty} p^{2t} \int_{\mathbb{Q}_p} \mathbb{1}_{\left(p^{-z}\mathbb{U}_p^n\right)} h\left(\overrightarrow{v}, p^{-2t}Q_o(\overrightarrow{v})\right) d\overrightarrow{v},
\end{aligned}$$

where $\mathbb{1}_A$ is a characteristic function of A and the last equality holds because the summation over z is actually a finite sum. Hence we may assume that

$$h = \mathbb{1}_A \cdot \mathbb{1}_B \text{ for } A \subseteq p^{-z}\mathbb{U}_p^n \text{ and } B \subseteq p^{-z_0}\mathbb{Z}_p.$$

Since such a function h is approximated by characteristic functions of subsets induced from the projection $\pi_k : p^{-z}\mathbb{Z}_p^n(p^{-z_0}\mathbb{Z}_p) \to p^{-z}\mathbb{Z}_p^n/p^k\mathbb{Z}_p^n(p^{-z_0}\mathbb{Z}_p/p^k\mathbb{Z}_p^n)$ so that we further assume that there is $k \gg 0$ such that

$$\begin{aligned}
\overrightarrow{v} \in A &\Leftrightarrow \overrightarrow{v} + p^k\mathbb{Z}_p^n \subset A, \\
\zeta \in B &\Leftrightarrow \zeta + p^k\mathbb{Z}_p \subset B.
\end{aligned}$$

Then

$$p^{2t}\int_{\mathbb{Q}_p^n} \mathbb{1}_A(\overrightarrow{v})\mathbb{1}_B(p^{-2t}Q_o(\overrightarrow{v}))d\overrightarrow{v}$$
$$= p^{2t}\int_{\mathbb{Q}_p^n} \mathbb{1}_{(\{\overrightarrow{v}\in\mathbb{Q}_p^n:\|\overrightarrow{v}\|=p^z,\overrightarrow{v}\in A,p^{-2t}Q_o(\overrightarrow{v})\in B\})}(\overrightarrow{v})d\overrightarrow{v}$$

Now consider $\overrightarrow{v}' = p^{-z}\overrightarrow{e}_1 + Q_o(\overrightarrow{v})p^z\overrightarrow{e}_n = p^{-z}\overrightarrow{e}_1 + x_n\overrightarrow{e}_n$ and choose $t_o \gg k$ so that if $t > t_o$ and $p^{-2t}Q_o(\overrightarrow{v}) \in B$, then

$$\|Q_o(\overrightarrow{v})p^z\| < p^z \text{ so that } \|\overrightarrow{v}'\| = \|\overrightarrow{v}\| = p^z.$$

By transitivity of K, there is $k \in K$ such that $k\overrightarrow{v} = \overrightarrow{v}'$. We claim that

$$dx_1\cdots dx_n = \mathrm{vol}(K.\overrightarrow{e}_1)p^{-2t-z}dm(k)d\zeta, \tag{6.13}$$

where $\zeta = p^{-2t}Q_o(\overrightarrow{v})$. For this, again by transitivity of K, it suffices to show that the equality (6.13) holds on the local domain $U = \{(x_1,\dots,x_n) : |x_1| = p^z, |x_i| < |x_1|, 2 \le i \le n\}$. Then the local coordinate of $K.\overrightarrow{v}'\cap U$ is given by

$$(x_1,x_2,\dots,x_{n-1},x_n) = \left(p^{-z}u, x_2,\dots,x_{n-1}, p^z u^{-1}(p^{2t}\zeta - Q_o(0,x_2,\dots,x_{n-1},0))\right),$$

where $u \in \mathbb{U}_p$. By change of variables,

$$dx_1\cdots dx_n = p^{-2t-z}dx_1\cdots dx_{n-1}d\zeta$$

and we have the equality

$$dx_1\cdots dx_{n-1}d\zeta = \mathrm{vol}(K.\overrightarrow{e}_1)dm(k)d\zeta,$$

which completes the claim. Therefore,

$$p^{2t}\int_{\mathbb{Q}_p^n} \mathbb{1}_{(\{\overrightarrow{v}\in\mathbb{Q}_p^n:\|\overrightarrow{v}\|_p=p^z,\overrightarrow{v}\in A,p^{-2t}Q_o(\overrightarrow{v})\in B\})}(\overrightarrow{v})d\overrightarrow{v} =$$

$$
\begin{aligned}
&= p^{2t}\int_{\mathbb{Q}_p^n} \mathbb{1}_{(\{p^{-z}k^{-1}\overrightarrow{e}_1+p^{2t}p^{z}k^{-1}\zeta\overrightarrow{e}_n : p^{-z}k^{-1}\overrightarrow{e}_1\in A,\zeta\in B\})}(\overrightarrow{v})d\overrightarrow{v} \\
&= p^{2t}\int_{\mathbb{Q}_p}\int_K \mathbb{1}(A)](p^{-z}\,k^{-1}\overrightarrow{e}_1)\cdot \mathbb{1}_B(\zeta)\left(\mathrm{vol}(K.(p^{-z}\overrightarrow{e}_1))p^{-2t-z}dm(k)d\zeta\right) \\
&= \mathrm{vol}(K.\overrightarrow{e}_1)\int_{\mathbb{Q}_p}\int_K \mathbb{1}_A(p^{-z}\,k^{-1}\overrightarrow{e}_1)\cdot \mathbb{1}_B(\zeta)dm(k)p^{z(n-2)}d\zeta,
\end{aligned}
$$

which shows the lemma. $\square$

6.2 $\mathcal{S}$-arithmetic Result

Recall that we denote $t = (t_0, t_1, \ldots, t_s) \in \mathbb{R}\times\mathbb{Z}^s$, $T = (e^{t_0}, p_1^{t_1}, \ldots, p_s^{t_s})$ and $|T| = e^{t_0}\prod_{p\in S_f} p^{t_p}$.

Theorem 6.2.1. Let $\mathcal{O}(\sigma) = \mathcal{O}(p, q, d(S_f), \varepsilon(S_f))$ denote the space of quadratic forms Q on $\mathbb{Q}_S^n$ such that

1. $\mathrm{sign}(Q_0) = (p, q)$ and the discriminant of $Q_0 = (-1)^q$,
2. $\mathrm{sign}(Q_j) = (d_j, \varepsilon_j)$ and the discriminant of Q_j is d_j.

Let $I(a, b)$ be a set of $\mathbb{Q}_S^n$ given by

$$I(a,b) = (a_0, b_0)\times\prod_{i=1}^{s}(a_i + p_i^{b_i}\mathbb{Z}_{p_i})$$

and $\mathcal{D}$ be a compact subset of $\mathcal{O}(\sigma)$. If the signature of real quadratic form is (p, q) with $p \geq 3$, then there exists a constant c depending only $\mathcal{D}$, $I(a, b)$ and Ω such that for any $Q \in \mathcal{D}$ and all $t \succ 0$,

$$\left|\mathbb{Z}_S^n \cap V^Q_{I(a,b)} \cap T\Omega\right| < c|T|^{n-2}.$$

Proof. We can take a compact set $C \subset G$ so that every quadratic form in $\mathcal{D}$ is of the form Q_0^g for some $g \in C$. Since C is compact, there exists β such that for every $g \in C$ and every $\overrightarrow{v} \in \mathbb{Q}_S^n$, $\beta^{-1}\|\overrightarrow{v}_\nu\|_\nu \leq \|\overrightarrow{v}_\nu\| \leq \beta\|\overrightarrow{v}_\nu\|_\nu$ for every $\nu \in S$. Let $\epsilon > 0$ be given and $g \in C$ be arbitrary. Choose a continuous non-negative function

$f = \prod_{\nu=0}^{n} f_\nu$ on $\mathbb{Q}_S^n$ of compact support so that $J_{f_\nu} \geq 1+\epsilon$ on $[\beta^{-1}, 2\beta] \times [a_0, b_0]$ if $\nu = \infty$ and $J_{f_\nu} \geq 1+\epsilon$ on $[\beta^{-1}, p\beta] \times (a_\nu + p^b\mathbb{Z}_p)$ if $p \in S_f$. Then if $\overrightarrow{v} \in \mathbb{Q}_S^n$ satisfies

$$\begin{aligned} e^{t_0} \leq \|\overrightarrow{v}_0\| \leq 2e^{t_0}, a_0 \leq Q_o^0(g_0\overrightarrow{v}_0) \leq b_0 \quad \text{if } \nu = \infty \text{ and} \\ \|\overrightarrow{v}_\nu\| = p^{t_p}, Q_o^\nu(g_p\overrightarrow{v}_p) \in a_p + p^{b_p}\mathbb{Z}_p \quad \text{if } p \in S_f, \end{aligned} \tag{6.14}$$

then $J_{f_0}(\|g_0\overrightarrow{v}_0\|e^{-t_0}, Q_o^0(g_0\overrightarrow{v}_0)) \prod_{p\in S_f} J_{f_p}(\|g_p\overrightarrow{v}_p\|^{-1}p^{t_p}, Q_o^\nu(g_p\overrightarrow{v}_p)) \geq 1+\epsilon$. By Lemma 6.1.1 and Lemma 3.6 in [6], for sufficiently large t,

$$c(K)|T|^{n-2}\int_K f(a_tkg\overrightarrow{v})dm(k) \geq 1$$

if $\overrightarrow{v}$ satisfies (6.14). By summing over $\overrightarrow{v} \in \mathbb{Z}_S^n$, we get that

$$\begin{aligned} &\left|\mathbb{Z}_S^n \cap V_{I(a,b)}^Q \cap \left([e^{t_0}, 2e^{t_0}]S^{n-1} \times \prod_{p\in S_f} p^{-t_p}\mathbb{Z}_p \setminus p^{-t_p+1}\mathbb{Z}_p\right)\right| \\ &\qquad \leq \sum_{\overrightarrow{v}\in\mathbb{Z}_S} c(K)|T|^{n-2}\int_K f(a_tkg\overrightarrow{v})dm(k) = c(K)|T|^{n-2}\int_K \tilde{f}(a_tkg)dm(k). \end{aligned}$$

□

Proposition 6.2.2. Let $f = \prod_{\nu\in S} f_\nu$ be a product of non-negative continuous functions f_ν such that the support of f_ν is on $\mathbb{Q}_{\nu,+}^n$ and f_ν satisfies the property (6.1) when $\nu \in S_f$. Let $\nu = \prod_{\nu\in S}\nu_\nu$ be a product of positive continuous functions ν_ν such that for each $\nu \in S$, ν_ν is defined on $\{\overrightarrow{x} \in \mathbb{Q}_\nu^n : \|\overrightarrow{x}\|_\nu = 1\}$. Suppose that the signature of real quadratic form is (p, q) with $p \geq 3$. Then for given $\epsilon > 0$ and every $g \in G$, there exists a positive $t_o = (t_o^0, t_o^1, \ldots, t_o^s) \succ 0$ so that for $t > t_o$,

$$\left| |T|^{-(n-2)} \sum_{\overrightarrow{v}\in\mathbb{Z}_S^n} J_f(g, \overrightarrow{v}, t)\nu(g, \overrightarrow{v}) - c(K)\int_K \tilde{f}(a_tkg)\nu(k^{-1}\overrightarrow{e}_1)dm(k)\right| \leq \epsilon,$$

where $c(K) = \prod_{\nu\in S} c(K_\nu)$ and

$$J_f(g, \vec{v}, t) = J_{f_0}(\|g_0\vec{v}_0\|e^{-t_0}, Q_o^0(g_0\vec{v}_0)) \times \prod_{p\in S_f} J_{f_p}(p^t/\|g_p\vec{v}_p\|_p, Q_o^p(g_p\vec{v}_p)),$$

$$\nu(g, \vec{v}) = \nu_0(g_0\vec{v}_0/\|g_0\vec{v}_o\|) \times \prod_{p\in S_f} \nu_p(\|g_p\vec{v}_p\|_p g_p\vec{v}_p).$$

Proof. Since $J_f = \prod_{\nu\in S} J_{f_\nu}$ has compact support, by Theorem 6.2.1 for given $g \in G$, the number of $\vec{v} \in \mathbb{Z}_S^n$ such that $J_f(g, \vec{v}, t)\nu(g, \vec{v}) \neq 0$ is bounded by $c|T|^{n-2}$ for some $c > 0$. Moreover, since supremum norms $\|f_\nu\|_\infty$ and $\|J_{f_\nu}\|_\infty$ for each $\nu \in S$ are bounded, by Lemma 6.1.1 and Lemma 3.6 in [6], there exists $T_o \succ 0$ such that

$$\left| J_f(g, \vec{v}, t)\nu(g, \vec{v}) - c(K)|T|^{n-2} \int_K \tilde{f}(a(t)kg)\nu(k^{-1}\vec{e}_1)dm(k) \right| < \frac{\epsilon}{c}.$$

Therefore,

$$\left| |T|^{-(n-2)} \sum_{\vec{v}\in\mathbb{Z}_S^n} J_f(g, \vec{v}, t)\nu(g, \vec{v}) - c(K) \int_K \tilde{f}(a(t)kg)\nu(k^{-1}\vec{e}_1)dm(k) \right|$$
$$\leq |T|^{-(n-2)} \cdot \epsilon/c \cdot c|T|^{n-2} = \epsilon.$$

□

Lemma 6.2.3. Suppose $f = f_0 \times \prod_{p\in S_f} f_p$ be a continuous function of compact support on $\mathbb{Q}_{S,+}^n$, $\nu = \nu_0 \times \prod_{p\in S_f} \nu_p$ a non-negative continuous function on $\mathbb{S}^{n-1} \times \prod_{p\in S_f} \mathbb{U}_p$ defined in Proposition 6.1.1. Let h be a product $h_0 \times \prod_{p\in S_f} h_p$ of functions, where $h_0 = J_{f_0}(\|\vec{v}_0\|, \zeta_0)\nu_0(\vec{v}_0/\|\vec{v}_0\|)$ and $h_p = J_{f_p}(\|\vec{v}_p\|, \zeta_p)\nu_p(\|\vec{v}_p\|\vec{v}_p)$. Then

$$\frac{1}{|T|^{n-2}} \int_{\mathbb{R}^n} h_0(\vec{v}_0/T, Q_o^0(\vec{v}_0))d\vec{v}_0 \prod_{p\in S_f} \int_{\mathbb{Q}_p^n} h_p(p^{t_p}\vec{v}_p, Q_o^p(\vec{v}_p))d\vec{v}_p \tag{6.15}$$
$$\to c_{p,q} \prod_{p\in S_f} c(K_p) \int_{G/\Gamma} \tilde{f} \int_{K_0} \nu_0(k_0^{-1}\vec{e}_1)dm(k_0) \prod_{p\in S_f} \int_{K_p} \nu_p(k_p^{-1}\vec{e}_1)dm(k_p)$$

as T goes to infinity.

Proof. After applying the S-arithmetic version of Siegel integral formula, it suffices to show the following of the p-adic version of the proof of Lemma 3.9 in [6]. Since for any unit u,

$$\int_{\mathbb{Q}_p^{n-1}} f(p^{-r}, x_2, \ldots, x_{n-1}, x_n) dx_2 \cdots dx_n = \int_{\mathbb{Q}_p^{n-1}} f(p^{-r}u, x_2, \ldots, x_{n-1}, x_n) dx_2 \cdots dx_n,$$

we have that

$$\begin{aligned}\int_{\mathbb{Q}_p^n} f dx_1 \cdots dx_n &= \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p^{n-1}} f(x_1, x_2, \ldots x_{n-1}, x_n) dx_2 \cdots dx_n \cdot dx_1 \\ &= \sum_{r \in \mathbb{Z}} \left\{ \int_{\mathbb{Q}_p^{n-1}} f(p^{-r}, x_2, \ldots x_{n-1}, x_n) dx_2 \cdots dx_n \right\} \cdot p^r (1 - \frac{1}{p}).\end{aligned}$$

Since $p^{-r}x_n + Q_0(0, x_2, \ldots, x_{n-1}, 0) = \zeta$ and $p^r dx_n = d\zeta$ so that

$$\begin{aligned}\int_{\mathbb{Q}_p^n} f dx_1 \cdots dx_n &= \sum_{r \in \mathbb{Z}} \int_{\mathbb{Q}_p} \int_{\mathbb{Q}_p^{n-2}} f(p^{-r}, x_2, \ldots, x_{n-1}, x_n) dx_2 \cdots dx_{n-1} (1 - 1/p) d\zeta \\ &= \sum_{r \in \mathbb{Z}} \int_{\mathbb{Q}_p} J_f(p^{-r}, \zeta) p^{r(n-2)} (1 - 1/p) d\zeta.\end{aligned}$$

This and Proposition 6.1.2 show (6.15). □

Lemma 6.2.4. Let Ω be a set defined in (1.13). Then there exists a constant $\lambda = \lambda_{Q,\Omega}$ so that as $T \to \infty$,

$$\mathrm{vol}(V^Q_{I(a,b)} \cap T\Omega) \sim \lambda_{Q,\Omega} \mu(I(a,b)) |T|^{n-2},$$

where μ is the Haar measure on $\mathbb{Q}_S$.

Proof. Since the set Ω is the product of $\Omega_v \subset \mathbb{Q}_v^n$ for each place $v \in S$, it suffices to show that for each $v \in S$,

$$\mathrm{vol}(pr_v(V^Q_{I(a,b)}) \cap T^v \Omega_v) \sim \lambda_{Q^v, \Omega_v} \mathrm{vol}(pr_v(I(a,b)))(T^v)^{n-2}$$

The real case follows from Lemma 3.8 (ii) in [6]. Suppose $v = p \in S_f$. We want to prove that

$$\mathrm{vol}(\{\vec{v} \in \mathbb{Q}_p^n : Q(\vec{v}) \in a + p^b\mathbb{Z}_p\} \cap p^{-t}\Omega_p) \sim \lambda_{Q^p,\Omega_p} p^{-b} p^{n-2} \tag{6.16}$$

as $t \to \infty$. Recall that $\Omega_p = \{\vec{v} \in \mathbb{Q}_p^n : \|\vec{v}\| \le \rho_p(\|\vec{v}\|\vec{v})\}$, where $\rho(u\vec{v}) = \rho(\vec{v})$ for any $\vec{v} \in \mathbb{U}_p^n$ and $u \in \mathcal{U}_p$. We know that there is an element $g \in \mathrm{SL}_n(\mathbb{Q}_p)$ such that $Q^p(\vec{v}) = Q_o^p(g\vec{v})$. By changing $g^{-1}\Omega_p$ instead of Ω, we may assume that $Q^p = Q_o^p$. Note that ρ' which determines $g^{-1}\Omega_p$ also satisfies that $\rho'(\vec{v}) = \rho'(u\vec{v})$ for $u \in \mathcal{U}_p$.

Let $\mathbb{1}_{\hat{\Omega}_p \times (a+p^b\mathbb{Z}_p)}$ be the characteristic function where

$$\hat{\Omega}_p = \{\vec{v} \in \mathbb{Q}_p^n : \|\vec{v}\| = \rho(\|\vec{v}\|\vec{v})\}.$$

We now apply Proposition 6.1.2 to sequences (h_m), (h'_m) of compactly supported continuous positive functions such that $h_m \le \mathbb{1}_{\hat{\Omega}_p \times (a+p^b\mathbb{Z}_p)} \le h'_m$ and $|h_m - h'_m|_\infty \to 0$ as $m \to \infty$. Since h_m, h'_m are compactly supported, by definition, $\lim_{m\to\infty} L_1(h_m) = \lim_{m\to\infty} L_1(h'_m)$. Hence

$$\begin{aligned}
&\lim_{t\to\infty} p^{-(n-2)t}\,\mathrm{vol}(\{\vec{v} \in \mathbb{Q}_p^n : Q(\vec{v}) \in a + p^b\mathbb{Z}_p\} \cap p^{-t}\hat{\Omega}_p) \\
&\qquad = \lim_{t\to\infty} p^{-(n-2)t} \int_{\mathbb{Q}_p^n} \mathbb{1}_{\hat{\Omega}_p \times (a+p^b\mathbb{Z}_p)}(p^t\vec{v}, Q(\vec{v}))\,d\vec{v} \\
&\qquad = \mathrm{vol}(K.\vec{e}_1)p^{-b} \sum_{z=-\infty}^{\infty} p^{(n-2)z} \int_K \mathbb{1}_{\hat{\Omega}}(p^{-z}k\vec{e}_1)\,dm(k) = \lambda_{Q,\hat{\Omega}_p} p^{-b}.
\end{aligned}$$

By considering each $p^i\hat{\Omega}$ instead of $\hat{\Omega}$, $i \in \mathbb{N}$ and adding these quantities, we can find a constant $\lambda_{Q,\Omega}$ satisfying (6.16). □

Theorem 6.2.5. [15] Let Q be a quadratic form such that the signature of the real quadratic form is (p, q) with $p \ge 3$. Let ϕ be a continuous compactly supported function on G/Γ and ν be a positive function on $\mathbb{S}^{n-1} \times \prod_{p \in S_f} \mathbb{U}_p^n$. Let $\mathcal{D}$ be any compact set in G/Γ. Then for any $\epsilon > 0$, there exist finitely many points $x_1, \ldots, x_\ell \in G/\Gamma$ such that

1. the orbit $Hx_1, \ldots, Hx_\ell$ are closed and have finite H-invariant measures,

2. for any compact set $\mathcal{F} \subseteq \mathcal{D} - \cup_{i=1}^{\ell} Hx_i$, there exists $t_o \in \mathbb{R}_{>0} \times \mathbb{N}^s$ so that for any $x \in \mathcal{F}$ and $t > t_o$,

$$\left| \int_K \phi(a_t kx)\nu(k)dm(k) - \int_{G/\Gamma} \phi dg \int_K \nu dm \right| \leq \epsilon. \tag{6.17}$$

Define

$$A(r) = \{x \in G/\Gamma : \alpha(x) \leq r\}. \tag{6.18}$$

Using S-adic version of Mahler's compactness criterion(See [12, Subsection 7.10], we obtain that $A(r)$ is compact for any $r > 0$.

Theorem 6.2.6. Let Q be a quadratic form such that the signature of the real quadratic form is (p, q) with $p \geq 3$. Then Theorem 6.2.5 holds for a continuous function ϕ on G/Γ such that for some s, $0 \leq s \leq 2$, there exists a constant $C > 0$ so that

$$|\phi(\Delta)| < C\alpha(\Delta)^s$$

for all Δ in G/Γ.

Proof. We may assume that ϕ is nonnegative. For each $r \in \mathbb{R}_{>0}$, we choose a continuous function g_r on G/Γ such that $0 \leq g_r(x) \leq 1$, $g_r(x) = 1$ if $x \in A(r)$ and $g_r(x) = 0$ outside $A(r+1)$. Then

$$\phi(a_t kx) = (g_r\phi)(a_t kx) + ((1 - g_r)\phi)(a_t kx).$$

Following the proof of Theorem 3.5 in [6], let $\beta = 2-s$. Since $((1-g_r)\phi)(y) = 0$ if $y \in A(r)$,

$$\begin{aligned} ((1 - g_r)\phi)(y) &\leq C(1 - g_r)(y)\alpha(y)^{2-\beta} \leq C\alpha(y)^{2-\beta/2}(1 - g_r)(y)\alpha(y)^{-\beta/2} \\ &\leq Cr^{-\beta/2}\alpha(y)^{2-\beta/2}. \end{aligned}$$

By Theorem 5.3.5 and the fact that $\|\nu\|_\infty < \infty$, there exists $C' > 0$ such that

$$\begin{aligned}\int_K ((1-g_r)\phi)(a_t k x)\nu(k)dm(k) &\le C r^{-\beta/2}\int_K \alpha(a_t k x)^{2-\beta/2}\nu(k)dm(k) \\ &\le C' r^{-\beta/2}\end{aligned} \tag{6.19}$$

On the other hand, since $g_r\phi$ is compactly supported, we can deduce from Theorem 6.2.5 that for sufficiently large $t \in \mathbb{R}_{>0} \times \mathbb{N}^s$,

$$\left|\int_K (g_r\phi)(a_t k x)\nu(k)dm(k) - \int_{G/\Gamma}(g_r\phi)(y)dg(y)\int_K \nu(k)dm(k)\right| < \epsilon/2, \tag{6.20}$$

where dg is the normalized Haar measure on G/Γ.

Since $g_r\phi \to \phi$ as $r \to \infty$, (6.19) and (6.20) show the theorem. □

6.3 Proof of Main Theorem(Theorem 1.0.7)

For a compactly supported continuous function h on $(\mathbb{Q}_S^n - \{\overrightarrow{0}\}) \times \mathbb{Q}_S$, define

$$L(h) = \lim_{T\to\infty} |T|^{-(n-2)} \int_{\mathbb{Q}_S^n} h_0(e^{-t}\overrightarrow{v}_0, Q_o^0(\overrightarrow{v}_0)) \times \prod_{p\in S_f} h_p(p^t\overrightarrow{v}_p, Q_o^p(\overrightarrow{v}_p))d\overrightarrow{v}.$$

Before starting the proof, consider the space $\mathcal{L}$ of finite linear combinations of functions of the form

$$J_{f_0}(\|\overrightarrow{v}_0\|, \zeta_0)\nu_0(\overrightarrow{v}_0/\|\overrightarrow{v}_0\|) \times \prod_{p\in S_f} J_{f_p}(\|\overrightarrow{v}_p\|, \zeta_p)\nu(\|\overrightarrow{v}_p\|\overrightarrow{v}_p),$$

where f_0, f_p are continuous compactly supported functions on $\mathbb{R}_+^n$, $\mathbb{Q}_{p,+}^n$, $p \in S_f$ respectively and ν_0, ν_p are non-negative continuous function on $\mathbb{S}^{n-1}$, $\mathbb{U}_p^n$ respectively. Note that the function $J_{f_0} \times \prod_{p\in S_f} J_{f_p}$ has the same values on the orbit $K.\overrightarrow{v}$ for each $\overrightarrow{v} \in \mathbb{Q}_S^n$. By letting $r_\nu = \|\overrightarrow{v}_\nu\|$, we may consider $J_{f_0} \times \prod_{p\in S_f} J_{f_p}$ as functions defined over $(\mathbb{R}\times\mathbb{Z}^s)\times\mathbb{Q}_S$ and we can easily deduce that any continuous compactly supported functions on $(\mathbb{R}\times\mathbb{Z}^s)\times\mathbb{Q}_S$ are approximated by $J_{f_0}\times\prod_{p\in S_f} J_{f_p}$ for some functions f_ν's from Stone-Weierstrass Theorem(p-adic case, see [5]). It implies that for each $x \in \mathcal{D}$, there are h_x, h'_x such that for all $\overrightarrow{v} \in \mathbb{Q}_S^n$ and $\zeta \in \mathbb{Q}_S$,

$$h_x(x\overrightarrow{v}, \zeta) \leq \mathbb{1}_{\hat{\Omega}} \leq h'_x(x\overrightarrow{v}, \zeta) \tag{6.21}$$

and

$$|L(h_x) - L(h'_x)| < \epsilon. \tag{6.22}$$

where $\mathbb{1}_{\hat{\Omega}}$ is the characteristic function of

$$\hat{\Omega} = \left\{ \overrightarrow{v} \in \mathbb{Q}_S^n : \begin{array}{c} \rho_0(\overrightarrow{v}_0/\|\overrightarrow{v}_0\|)/2 < \|\overrightarrow{v}_0\| < \rho_0(\overrightarrow{v}_0/\|\overrightarrow{v}_0\|) \text{ and} \\ \|\overrightarrow{v}_i\| = \rho_i(\|\overrightarrow{v}_i\|\overrightarrow{v}_i), 1 \leq i \leq s \end{array} \right\} \tag{6.23}$$

By Proposition 6.2.2, theorem 6.2.6 and Proposition 6.2.3, there exist points $x_1, \ldots, x_\ell \in G/\Gamma$ so that Hx_i is closed, $1 \leq i \leq \ell$, and for each compact subset $\mathcal{F}$ of $\mathcal{D} - \cup_{i=1}^{\ell} Hx_i$, there exists $t_o \in \mathbb{R}_{>0} \times \mathbb{N}^s$ so that for all $t > t_o$ and $x = g\Gamma \in \mathcal{F}$,

$$\begin{aligned}
&\left| |T|^{-(n-2)} \sum_{\overrightarrow{v} \in \mathbb{Z}_S^n} h_0(g\overrightarrow{v}_0 e^{-t_0}, Q_o^0(g_0\overrightarrow{v})) \prod_{p \in S_f} h_p(p^{t_p} g\overrightarrow{v}_p, Q_o^p(g\overrightarrow{v}_p)) - L(h) \right| \\
&\leq \Bigg| |T|^{-(n-2)} \sum_{\overrightarrow{v} \in \mathbb{Z}_S^n} h_0(g\overrightarrow{v}_0 e^{-t_0}, Q_o^0(g_0\overrightarrow{v})) \prod_{p \in S_f} h_p(p^{t_p} g\overrightarrow{v}_p, Q_o^p(g\overrightarrow{v}_p)) \\
&\qquad\qquad - c(K)\int_K \tilde{f}(a_t kx)\nu(k^{-1}\overrightarrow{e}_1)dm(k) \Bigg| \\
&+ \left| c(K)\int_K \tilde{f}(a_t kx)\nu(k^{-1}\overrightarrow{e}_1)dm(k) - c(K)\int_{G/\Gamma} \tilde{f}dg \int_K \nu dm \right| + \\
&\left| c(K)\int_{G/\Gamma} \tilde{f}dg \int_K \nu dm - |T|^{n-2} \int_{\mathbb{Q}_S^n} h_0(e^{-t}\overrightarrow{v}_0, Q_o^0(\overrightarrow{v}_0)) \times \prod_{p \in S_f} h_p(p^t\overrightarrow{v}_p, Q_o^p(\overrightarrow{v}_p))d\overrightarrow{v} \right| < \epsilon,
\end{aligned} \tag{6.24}$$

and by definition of $L(h)$, we can further assume that

$$\left| |T|^{-(n-2)} \int_{\mathbb{Q}_S^n} h_0(e^{-t}\overrightarrow{v}_0, Q_o^0(\overrightarrow{v}_0)) \times \prod_{p \in S_f} h_p(p^t \overrightarrow{v}_p, Q_o^p(\overrightarrow{v}_p)) d\overrightarrow{v} - L(h) \right| \leq \epsilon. \tag{6.25}$$

By approximating argument with h_x, h'_x with a suitable ϵ, for every $\theta > 0$, there exist finitely many points $x_1, \ldots, x_\ell$ with Hx_i closed and for an arbitrary compact subset $\mathcal{F} \subseteq \mathcal{D} - \cup_{i=1}^{\ell} Hx_i$, there exists $t_o \in \mathbb{R}_{>0} \times \mathbb{N}^s$ such that for every $x = g\Gamma \in \mathcal{F}$ and every $t > t_o$, we obtain

$$(1-\theta)\,\mathrm{vol}(V^Q_{I(a,b)} \cap T\hat{\Omega}) \leq |\mathbb{Z}_S^n \cap \mathrm{vol}(V^Q_{I(a,b)} \cap T\hat{\Omega}| \leq (1+\theta)\,\mathrm{vol}(V^Q_{I(a,b)} \cap T\hat{\Omega}).$$

By the argument of geometric series and Lemma 6.2.4, it follows that for sufficiently large T, for every quadratic form Q corresponding $x \in \mathcal{F}$,

$$|\mathbb{Z}_S^n \cap V^Q_{I(a,b)} \cap T\Omega| \sim \lambda_{Q,\Omega}\mu(I(a,b))|T|^{n-2}.$$

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국문초록

이 논문에서 우리는 동질 역학에서 특정 조건을 가진 비유계 함수에 대한 균질 분포 정리를 이용하여 이차 형식들의 기하 구조를 공부한다. 첫번째 주제는 임의의 중심점을 가지고 유계 계수를 가지는 타원체와 관련된 지수 합의 이차 평균값에 대해 연구이다. 일반적인 경우에 대해서는 하한 극한을 구할 수 있고, 만일 중심점이 디오판틴 타입일 경우 하한 극한이 상한 극한과 일치함을 보일 수 있었다. 이 결과는 마크로프(Marklof)의 연구를 일반화한 결과이다.

두번째 주제는 S-산술 이차 형식에 대하여 정량적 오펜하임 추측을 일반화하였다. $\mathbb{Q}_S$의 임의의 열린 집합 I에 대하여, 우리는 노움 T를 가지고 이차 형식의 함수값이 I에 포함되는 S-정수값을 가지는 벡터들의 갯수가 T가 무한대로 증가할 때 $c(Q, I)T^{n-2}$로 근사한다는 것을 보일 수 있었다. 이 연구는 에스킨(Eskin), 마구리스(Margulis) 그리고 모제스(Mozes)의 실수에서의 정량적 오펜하임 추측에 대한 결과를 일반화하였다.

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